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Consistency of the structured total least squares estimator in a multivariate errors-in-variables model

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Abstract

The structured total least squares estimator, defined via a constrained optimization problem, is a generalization of the total least squares estimator when the data matrix and the applied correction satisfy given structural constraints. In the paper, an affine structure with additional assumptions is considered. In particular, Toeplitz and Hankel structured, noise free and unstructured blocks are allowed simultaneously in the augmented data matrix. An equivalent optimization problem is derived that has as decision variables only the estimated parameters. The cost function of the equivalent problem is used to prove consistency of the structured total least squares estimator. The results for the general affine structured multivariate model are illustrated by examples of special models. Modification of the results for block-Hankel/Toeplitz structures is also given. As a by-product of the analysis of the cost function, an iterative algorithm for the computation of the structured total least squares estimator is proposed.

MSC: 62F12; 93E12; 62J05; 65F20; 62H12

Keywords: Block–Hankel/Toeplitz structure; Consistency; Dynamic errors-in-variables model; Iterative algorithm; Structured total least squares; Total least squares

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1. Introduction

The *total least squares* (TLS) problem (Golub and Van Loan, 1980; Van Huffel and Vandewalle, 1991),

$$\min_{\Delta A, \Delta B, X} \| [\Delta A \ \ \Delta B] \|_{\mathrm{F}}^2 \quad \text{s.t.} \quad (A + \Delta A)X = B + \Delta B \tag{1.1}$$

is a generalization of the ordinary least squares method when an errors-in-variables (EIV) model

$$A = A_0 + \tilde{A}, \quad B = B_0 + \tilde{B}, \quad A_0 X_0 = B_0 \tag{1.2}$$

is considered. Here $\tilde{A}$, $\tilde{B}$ are measurement errors and A_0, B_0 are true values, that satisfy the linear model for some unknown true value X_0 of the parameter X. The TLS estimate of X_0, i.e., the solution of (1.1), corresponding to X, is proven to provide a consistent estimate of X_0, when the elements of $\tilde{A}$ and $\tilde{B}$ are zero mean i.i.d. The *generalized total least squares* (GTLS) problem (Van Huffel and Vandewalle, 1989) extends consistency of the TLS estimator to cases where the errors $[\tilde{A} \ \ \tilde{B}]$ are zero mean, row-wise independent, and with equal row covariance matrix, known up to a factor of proportionality. Efficient and reliable algorithms, based on the (generalized) singular value decomposition, exist for the computation of the TLS and the GTLS solutions.

A further generalization for the case when the rows of $[\tilde{A} \ \ \tilde{B}]$ have different covariance matrices (but are still mutually independent) is the *element-wise weighted total least squares* (EW-TLS) estimator (De Moor, 1993; Premoli and Rastello, 2002). Consistency of the EW-TLS estimator is proven in Kukush and Van Huffel (2004). The EW-TLS problem is a difficult non-convex optimization problem and its solution cannot be found in terms of the singular value decomposition of the data matrix. An iterative optimization procedure for its solution is proposed in Premoli and Rastello (2002) and Markovsky et al. (2002a).

In De Moor and Roorda (1994) the so-called *dynamic total least squares* problem is considered. The problem formulation in De Moor and Roorda (1994) is parallel to this of the TLS problem but a discrete-time linear dynamical model is postulated instead of the static model $A_0 X_0 = B_0$. The equations of the dynamical model over a finite time horizon can be written as a linear system of equations $A_0 X_0 = B_0$ with A_0, a structured matrix, e.g., Toeplitz or Hankel matrix. This gives rise to a TLS-type problem with the additional constraint that the correction matrix ΔA obeys a certain known structure.

The resulting problem is called a *structured total least squares* (STLS) problem. In De Moor (1993) a list of applications of the STLS problem is given. Among them we mention a single-input single-output identification problem, an H_2-approximation problem, and an errors-in-variables version of the Kalman filter. The TLS and GTLS problems are special cases of the STLS problem. Due to the structure assumption, the errors in the STLS problem are correlated among the rows and in this respect the STLS problem is more general than the EW-TLS problem. For the consistency of the STLS estimator, however, we assume stationarity of the errors. Such an assumption is not enforced in the framework of the EW-TLS problem, so that the EW-TLS problem is not a special case of the STLS problem formulation considered in this paper.

The STLS problem fits within the Markov framework for semi-linear models of Pintelon and Schoukens (2001, Chapter 7), i.e., models linear-in-observations and (non)linear-in-the-model-parameters. In the STLS problem, however, there is a structure assumption on the true values A_0 and B_0, while in the semi-linear model, a structure assumption is imposed only on the errors $\tilde{A}$ and $\tilde{B}$. Moreover in Pintelon and Schoukens (2001) the parameter set is assumed to be compact and the errors to be normally distributed, while in the present paper, the parameter set is closed but not necessary bounded and the error distribution is not necessary Gaussian.

Although the STLS problem is a very general modeling framework, its computation is also a difficult non-convex optimization problem. An overview of algorithms for STLS computation is given in Lemmerling (1999, Section 4), and numerically efficient algorithms based on the Generalized Schur Algorithm are developed in Mastronardi et al. (2000), Lemmerling et al. (2000) and Mastronardi (2001).

The main contribution of the present paper is a proof of statistical consistency of the STLS estimator. Results on the consistency of the STLS estimate are presented for an affine structured multivariate EIV model. The proofs are similar to the ones presented in Kukush and Van Huffel (2004) for the EW-TLS estimator, but the presence of the structural relations makes the consistency proofs more complicated.

Most of the statistical literature of EIV modeling is devoted to unstructured problems, e.g., the classical book on measurement error models (Fuller, 1987) does not treat structured EIV models. Special cases of the STLS consistency problem are considered in the system identification literature. We mention the papers of Aoki and Yue (Aoki and Yue, 1970a, b), where consistency of the maximum likelihood estimator for an auto regressive moving average (ARMA) model is proven. Their estimator is a special instance of the STLS estimator of this paper when the structure of the extended data matrix $[A\ \ B]$ is a Hankel matrix next to another Hankel matrix, see Section 12.2. Estimators, different from the STLS one, are proven to be consistent for the dynamic EIV model. They are, however, statistically less efficient than the STLS estimator. Among them we mention the weighted GTLS estimator and bootstrapped TLS estimators (Pintelon et al., 1998), and the bias corrected least squares estimator (Stoica and Söderström, 1982). The consistency properties of the STLS estimator in the generality of our formulation have not been considered previously in the literature.

As a by-product of the analysis, we propose an algorithm similar to the one proposed for the EW-TLS problem (Markovsky et al., 2002a). In a companion paper (Markovsky et al., 2004), we implemented the proposed algorithm and compare it with other existing methods, e.g., the methods of Lemmerling (1999) and Mastronardi (2001). In terms of computational efficiency, our proposal is competitive with the fast methods of Mastronardi (2001).

The notation we use is standard: $\mathbb{R}$ denotes the set of the real numbers, $\mathbb{C}$ the set of the complex numbers, $\mathbb{Z}$ the set of the integer numbers, and $\mathbb{N}$ the set of the natural numbers. Any $p \times q$ matrix A is defined by $[a_{ij}]_{i=1,\dots,p}^{j=1,\dots,q}$, where a_{ij} denotes the $(i,\ j)$th element of A. We denote the transpose of the rows of A, by a_i, i.e., $A^\top = [t_1 \cdots t_q]$. $\|x\|$ is the Euclidean norm of the vector x and $\|A\|_{\mathrm{F}}$ is the Frobenius norm of the matrix A. The notation $\sigma(A)$ is used for the spectrum of the operator A, A^* is the adjoint operator, and $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) is the minimum (maximum) eigenvalue of a symmetric matrix A. For $\eta \in \mathbb{C}$, $\bar{\eta}$ is the complex conjugate of η. The bold symbol $\mathbf{E}$ denotes mathematical expectation and the bold symbol $\mathbf{P}$ denotes probability of an event, $\operatorname{cov}(\cdot)$ denotes the variance–covariance matrix of a vector of random variables, $\mathrm{O_p}(1)$ denotes a sequence of stochastically bounded random variables, and $\mathrm{o_p}(1)$ denotes a sequence of random variables that converges to zero in probability. In the formulas "const" denotes *any* constant value (for example, we can write $\text{const}^2 = \text{const}$). For two sets $\mathscr{S}_1$ and $\mathscr{S}_2$, $\mathscr{S}_1 \backslash \mathscr{S}_2$ denotes the set difference of $\mathscr{S}_1$ relative to $\mathscr{S}_2$.

The paper is structured as follows. Section 2 defines the STLS estimator as a solution of an optimization problem. The decision variables are the parameter X, to be estimated, and the nuisance parameters describing the structure. Section 3 derives an equivalent optimization problem in which the nuisance parameters are eliminated. The cost function of the equivalent problem is of the form $Q(X) = r^\top \Gamma^{-1} r$, where the vector r is an affine function of X, and the elements of the weight matrix Γ are quadratic functions of X. In Section 4, we study the properties of the weight matrix Γ. Under our assumptions, it is a block banded matrix. In Section 5, we redefine the cost function Q and the weight matrix Γ as functions of the extended parameter $X_{\text{ext}} := \begin{bmatrix} X \\ -I \end{bmatrix}$. This modification simplifies the analysis. In Section 6, we study the properties of the inverse weight matrix Γ^{-1}. We establish exponential decay of the elements of Γ^{-1}, away from the main diagonal. This property is crucial for the consistency proofs. In Section 7, we state the main results—weak and strong consistency of the STLS estimator. In preparation for the proofs, in Section 8 we make a decomposition of the cost function. In Appendix A, bounds for the summands of the decomposition are derived. Section 9 gives the proofs of the main results. In Section 10, we propose an algorithm for the computation of the STLS estimator. Section 11 considers specific examples of the general STLS multivariate EIV problem and specializes the consistency results for these cases. In Section 12, we describe three applications of the STLS problem: FIR system impulse response estimation, ARMA model identification, and Hankel low-rank approximation. Section 13 describes the necessary modification of the assumptions in the paper for consistency in the case of block-Toeplitz/Hankel structures. Section 14 gives conclusions and Appendix B reminds facts from the theory of stochastic fields.

2. The multivariate STLS problem

We consider the model $AX \approx B$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times d}$ are *observations*, and $X \in \mathbb{R}^{n\times d}$ is a *parameter* of interest. We suppose that (1.2) holds for some $X_0 \in \mathbb{R}^{n\times d}$. The matrix X_0 is the *true value* of the parameter, A_0, B_0 are non-stochastic true values of A and B, respectively, and $\tilde{A}$, $\tilde{B}$ are *errors*. Looking for asymptotic results in the estimation of X, we fix the dimension of X, n and d, and let the number of measurements m increase. The measurements are represented by the rows of A and the rows of B.

Let

$$\delta_{ij} := \tilde{a}_{ij} \quad \text{if } i = 1, 2, \ldots, m \ \text{ and } \ j = 1, \ldots, n$$

and

$$\delta_{i,n+k} := \tilde{b}_{ik} \quad \text{if } i = 1, 2, \ldots, m \ \text{ and } \ k = 1, \ldots, d.$$

We introduce the following assumptions:

(i) The data matrix $[A\ \ B]$ has the following partitioning:

$$C := [A\ \ B] = [A_{\mathrm{f}}\ \ A_{\mathrm{s}}\ \ B_{\mathrm{s}}\ \ B_{\mathrm{f}}],$$

where $A_{\mathrm{f}} \in \mathbb{R}^{m\times n_{\mathrm{f}}}$, $A_{\mathrm{s}} \in \mathbb{R}^{m\times n_{\mathrm{s}}}$, $B_{\mathrm{s}} \in \mathbb{R}^{m\times d_{\mathrm{s}}}$, and $B_{\mathrm{f}} \in \mathbb{R}^{m\times d_{\mathrm{f}}}$, with $n_{\mathrm{f}} + n_{\mathrm{s}} = n$, $d_{\mathrm{f}} + d_{\mathrm{s}} = d$, $n_{\mathrm{s}} + d_{\mathrm{s}} \geqslant 2$. Respectively,

$$C_0 := [A_0\ \ B_0] = [A_{0\mathrm{f}}\ \ A_{0\mathrm{s}}\ \ B_{0\mathrm{s}}\ \ B_{0\mathrm{f}}] \quad \text{and} \quad \tilde{C} := [\tilde{A}\ \ \tilde{B}] = [\tilde{A}_{\mathrm{f}}\ \ \tilde{A}_{\mathrm{s}}\ \ \tilde{B}_{\mathrm{s}}\ \ \tilde{B}_{\mathrm{f}}]$$

with $\tilde{A}_{\mathrm{f}} = 0$, $\tilde{B}_{\mathrm{f}} = 0$.

Condition (i) means that the first n_{f} columns in A and the last d_{f} columns in B are *error-free*. An example with error-free columns is the ARMA model with noisy output but noise-free input, see Section 12.2. If $n_{\mathrm{f}} = 0$, then the block A_{f} is absent, and if $d_{\mathrm{f}} = 0$, then the block B_{f} is absent.

(ii) There is an a priori known affine function (the structure in the problem)

$$S : \mathbb{R}^{n_p} \to \mathbb{R}^{m\times(n_{\mathrm{s}}+d_{\mathrm{s}})}, \qquad S(p) := S_0 + \sum_{k=1}^{n_p} S_k p_k \ \text{ for all } \ p \in \mathbb{R}^{n_p},$$

with $md \leqslant n_p \leqslant m(n_{\mathrm{s}} + d_{\mathrm{s}})$, such that

$$C_{\mathrm{s}} := [A_{\mathrm{s}}\ \ B_{\mathrm{s}}] = S(p)$$

for some parameter vector $p \in \mathbb{R}^{n_p}$. Since the maximum number of parameter equals the total number of elements in C_{s}, we have that $n_p \leqslant m(n_{\mathrm{s}} + d_{\mathrm{s}})$.
The matrix $C_{0\mathrm{s}} := [A_{0\mathrm{s}}\ \ B_{0\mathrm{s}}]$ also satisfies the affine function S, i.e., $C_{0\mathrm{s}} = S(p_0)$, for some unknown parameter vector $p_0 \in \mathbb{R}^{n_p}$. The vector p is a noisy measurement of p_0, i.e., $p = p_0 + \tilde{p}$, where $\tilde{p}$ is a zero mean random vector with a positive definite variance–covariance matrix $V_{\tilde{p}}$.

(iii) All the errors δ_{ij} have zero mean and finite second moments, and the covariance structure of $[\delta_{ij}]_{i=1,\dots,m}^{j=1,\dots,n+d}$ is known up to a factor of proportionality.

We mention that due to assumption (iii), n_{f} and d_{f} are known. Let

$$A^\top =: [a_1 \cdots a_m], \qquad B^\top =: [b_1 \cdots b_m].$$

Similar notation is used for the rows of A_0, $\tilde{A}$, $\tilde{A}_{\mathrm{s}}$, etc., for example, $\tilde{B}_{\mathrm{s}}^\top =: [\tilde{b}_{\mathrm{s}1} \cdots \tilde{b}_{\mathrm{s}m}]$. Let $s := n_{\mathrm{s}} + d_{\mathrm{s}} - 1$.

(iv) The sequence $\{\tilde{c}_{\mathrm{s}i}^\top = [\tilde{a}_{\mathrm{s}i}^\top\ \ \tilde{b}_{\mathrm{s}i}^\top]^\top,\ i = 1, 2, \dots, m\}$ is stationary in a wide sense, and s-dependent.

A centered sequence $\{v_i,\ i = 1, 2, \dots\}$ of random (column) vectors is called *stationary in a wide sense* if $\mathbf{E} v_i v_{i+k}^\top$, $i = 1, 2, \dots$, $k = 0, 1, 2, \dots$, depends only on k and does not depend on i. A sequence of random vectors (or matrices) $\{v_i,\ i=1,2,\ldots\}$ is called *q-dependent*, $q \geqslant 1$, if for each i, the two sequences $\{v_1,\ldots,v_i\}$ and $\{v_{i+q+1}, v_{i+q+2},\ldots\}$ are independent from each other.

Condition (iv) holds if $C_{\mathrm{s}} = [C_{\mathrm{s}1} \cdots C_{\mathrm{s}q}]$, where each of the blocks $C_{\mathrm{s}1},\ldots,C_{\mathrm{s}q}$ separately has either Hankel or Toeplitz structure, and the errors $\tilde{C}_{\mathrm{s}1},\ldots,\tilde{C}_{\mathrm{s}q}$ are independent. (We may allow certain dependence for the structural parameters coming from separate blocks). The block-Hankel/Toeplitz structure does not satisfy (iv) and is treated in Section 13.

(v) The true value $X_0 \in \Theta_X \subset \mathbb{R}^{n\times d}$, where Θ_X is a known closed set.

Note that Θ_X need not be compact.

For $X \in \mathbb{R}^{n\times d}$, we define

$$X_{\mathrm{ext}} := \begin{bmatrix} X \\ -I_d \end{bmatrix} \quad \text{and let} \quad X_{\mathrm{ext}} =: \begin{bmatrix} X_{\mathrm{fA}} \\ X_{\mathrm{s}} \\ X_{\mathrm{fB}} \end{bmatrix}$$

according to the equality

$$CX_{\mathrm{ext}} = A_{\mathrm{f}} X_{\mathrm{fA}} + C_{\mathrm{s}} X_{\mathrm{s}} + B_{\mathrm{f}} X_{\mathrm{fB}}.$$

(vi) For each $X \in \Theta_X$, $\mathrm{rank}(X_{\mathrm{s}}) = d$.

In particular, under assumption (vi), $n_{\mathrm{s}} + d_{\mathrm{s}} \geqslant d$.

Under assumptions (i)–(vi), the STLS problem consists in finding the value $\hat{X}$ of the unknown matrix X_0 and the value $\Delta\hat{p}$ of the unknown errors $\tilde{p}$ that minimize the weighted sum of squared corrections and make the corrected model $[A_{\mathrm{f}},\ S(p-\Delta p),\ B_{\mathrm{f}}]X_{\mathrm{ext}} = 0$ hold

$$\min_{X\in\Theta_X,\ \Delta p\in\mathbb{R}^{n_p}} ||V_{\tilde{p}}^{-1/2}\Delta p||^2 \quad \text{s.t.} \quad [A_{\mathrm{f}},\ S(p-\Delta p),\ B_{\mathrm{f}}]X_{\mathrm{ext}} = 0. \tag{2.1}$$

We give the following definition of the STLS estimator.

Definition 1. The STLS estimator $\hat{X}$ of X_0 is a measurable value of X, which solves the optimization problem (2.1).

Remark 1. The STLS estimator $\hat{X}$ equals the maximum likelihood estimator in case of Gaussian errors.

Remark 2. It can happen that for certain random realizations problem (2.1) has no solution. In that case, we set $\hat{X} = \infty$. Later on under consistency assumptions, we will show that $\hat{X} = \infty$ with probability tending to zero.

3. The cost function for X

For $X \in \Theta_X$ fixed, consider the solution of (2.1) as a function of X, i.e., we consider the function

$$Q(X) := \min_{\Delta p \in \mathbb{R}^{n_p}} ||V_{\tilde{p}}^{-1/2} \Delta p||^2 \quad \text{s.t.} \quad [A_{\mathrm{f}},\ S(p - \Delta p),\ B_{\mathrm{f}}]X_{\mathrm{ext}} = 0. \tag{3.1}$$

Then the STLS problem (2.1) is equivalent to the minimization of $Q(X)$, over $X \in \Theta_X$,

$$\min_{X \in \Theta_X} \ Q(X). \tag{3.2}$$

Next, we obtain the cost function $Q(X)$. We minimize analytically over Δp. (If $\Delta p_{\min}$ is a minimizer of (3.1), then $Q(X) = \Delta p_{\min}^{\top} V_{\tilde{p}}^{-1} \Delta p_{\min}$.) Denote the residual $AX - B$ by R and let r be the vectorized form of $R^{\top}$, i.e.,

$$R := AX - B = CX_{\mathrm{ext}}, \quad r := \mathrm{vec}(R^{\top}) = \mathrm{vec}([r_1 \cdots r_m]) = \begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix} \in \mathbb{R}^{md \times 1}.$$

We use similar notation for the random part $\tilde{R} = R - \mathbf{E}R$ of the residual, i.e.,

$$\tilde{R} := \tilde{A}X - \tilde{B} = \tilde{C}X_{\mathrm{ext}}, \quad \tilde{r} := \mathrm{vec}(\tilde{R}^{\top}) = \mathrm{vec}([\tilde{r}_1 \cdots \tilde{r}_m]) := \begin{bmatrix} \tilde{r}_1 \\ \vdots \\ \tilde{r}_m \end{bmatrix} \in \mathbb{R}^{md \times 1}.$$

The constraint of (2.1) is linear in Δp

$$\begin{aligned} &[\,A_{\mathrm{f}},\ S(p - \Delta p),\ B_{\mathrm{f}}\,]X_{\mathrm{ext}} = 0 \leftrightarrow CX_{\mathrm{ext}} = \sum_{k=1}^{n_p} S_k X_{\mathrm{s}} \Delta p_k \\ &\leftrightarrow R^{\top} = \sum_{k=1}^{n_p} (S_k X_{\mathrm{s}})^{\top} \Delta p_k \Leftrightarrow \mathrm{vec}(R^{\top}) = \sum_{k=1}^{n_p} \mathrm{vec}((S_k X_{\mathrm{s}})^{\top}) \Delta p_k \Leftrightarrow r = G\Delta p, \end{aligned}$$

where

$$G := [\mathrm{vec}((S_1 X_{\mathrm{s}})^{\top}) \cdots \mathrm{vec}((S_{n_p} X_{\mathrm{s}})^{\top})] \in \mathbb{R}^{md \times n_p}.$$

We have to solve the following problem:

$$\min_{\Delta p} \Delta p^{\top} V_{\tilde{p}}^{-1} \Delta p \quad \text{s.t.} \quad G\Delta p = r. \tag{3.3}$$

Note that we need to have the constraint that $G\Delta p = r$ is solvable, if (3.3) is to be feasible. Assuming that G is full rank, we need to have at least md parameters, i.e., $n_p \geqslant md$. Then (3.3) is a least-norm problem. Its solution is given by

$$\Delta p_{\min} = V_{\tilde{p}} G^{\top} (G V_{\tilde{p}} G^{\top})^{-1} r$$

and the optimal value is

$$Q(X) = \Delta p_{\min}^{\top} V_{\tilde{p}}^{-1} \Delta p_{\min} = r^{\top}(G V_{\tilde{p}} G^{\top})^{-1} r.$$

We can write Q as

$$Q(X) = \sum_{i,j=1}^{m} r_i^{\top} M_{ij} r_j,$$

where $M_{ij} \in \mathbb{R}^{d\times d}$ is the i, jth block of the matrix $(G V_{\tilde{p}} G^{\top})^{-1}$. The cost function of the EW-TLS problem (Markovsky et al., 2002a) is of the same type but $M_{ij} = 0$ for $i \neq j$; equivalently the matrix $G V_{\tilde{p}} G^{\top}$ is block diagonal.

Next, we show that $G V_{\tilde{p}} G^{\top} = \mathbf{E}\tilde{r}\tilde{r}^{\top}$. We have $G V_{\tilde{p}} G^{\top} = \mathbf{E}(G\tilde{p})(G\tilde{p})^{\top}$. But $\tilde{r} = \mathrm{vec}(\tilde{R}) = G\tilde{p}$, so that

$$\Gamma := \mathbf{E}\tilde{r}\tilde{r}^{\top} = G V_{\tilde{p}} G^{\top} \tag{3.4}$$

and

$$Q(X) = r^{\top}\Gamma^{-1} r. \tag{3.5}$$

Under further conditions on the parameter set Θ_X, Γ will be non-singular, for all $X \in \Theta_X$.

Note 1. In the unidimensional TLS case, (3.4) becomes $||Ax - b||^2/(1 + ||x||^2)$, which is a well-known formula—it gives the sum of the squared orthogonal distances from the data points to the regression hyperplane.

Note 2. For the Markov estimator in the semi-linear model, the cost function has exactly form (3.5), see Pintelon and Schoukens (2001, Chapter 7). But in the semi-linear model, there are no structure assumptions on the true values A_0 and B_0, therefore (3.5) is not a consequence of the results on the Markov estimator.

We proved the following statement.

Theorem 1. *The STLS estimator $\hat{X}$ exists if and only if there exists a minimum of* (3.2), *and then $\hat{X}$ is a minimum point of Q.*

Note 3. In the sequel we will use the structure assumption only on the errors, but not on A_0 and B_0. Without any changes, all our results are valid for the corresponding STLS estimator also in the case where only the errors are structured, because in that case the STLS estimator is the Markov estimator with the same cost function, see Note 2.

4. Properties of the weight matrix Γ

Let

$$V_{ij} := \mathbf{E}(\tilde{c}_i \tilde{c}_j^{\top}), \quad \text{for } i, j = 1, 2, \ldots, m, \tag{4.1}$$

where $\tilde{c}_i^\top$ is the ith row of $\tilde{C}$, i.e., $\tilde{C}^\top =: [\tilde{c}_1 \cdots \tilde{c}_m]$. We have $\tilde{r}_i = X_{\text{ext}}^\top \tilde{c}_i$, so that the positive semidefinite matrix Γ consists of the blocks

$$\begin{aligned} F_{ij} &= \mathbf{E}\tilde{r}_i\tilde{r}_j^\top = \mathbf{E}(X_{\text{ext}}^\top \tilde{c}_i \tilde{c}_j^\top X_{\text{ext}}) = X_{\text{ext}}^\top \mathbf{E}(\tilde{c}_i\tilde{c}_j^\top) X_{\text{ext}} \\ &= X_{\text{ext}}^\top V_{ij} X_{\text{ext}} \in \mathbb{R}^{d\times d} \quad \text{for } i, j = 1, \ldots, m. \end{aligned}$$

Due to condition (iv), $V_{ij} = V_{i-j}$ is a function of the difference $i - j$, and $V_{ij} = 0$ for i and j, such that $|i - j| \geqslant n_{\text{s}} + d_{\text{s}}$. Consequently, $F_{ij} = F_{i-j}$, and $F_{ij} = 0$ if $|i - j| \geqslant n_{\text{s}} + d_{\text{s}}$. Recall that $s = n_{\text{s}} + d_{\text{s}} - 1$. The matrix Γ has the block-banded structure,

$$\Gamma = \begin{bmatrix} F_0 & F_{-1} & \cdots & F_{-s} & & 0 \\ F_1 & F_0 & \ddots & \ddots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & F_{-s} \\ F_s & \ddots & \ddots & \ddots & \ddots & \vdots \\ & \ddots & \ddots & \ddots & F_0 & F_{-1} \\ 0 & & F_s & \cdots & F_1 & F_0 \end{bmatrix}, \tag{4.2}$$

where $F_k = F_{-k}^\top$ and $V_k = V_{-k}^\top$, $k = 0, 1, \ldots, s$.

In order to ensure that Γ is non-singular, we introduce the following assumption.

(vii) There exists $\lambda_0 > 0$, such that

$$v^\top \left(X_{\text{ext}}^\top \sum_{k=-s}^{s} V_k \omega^k X_{\text{ext}} \right) v \geqslant \lambda_0 ||X_{\text{s}} v||^2 \tag{4.3}$$

for all $X \in \Theta_X$, $v \in \mathbb{R}^{d\times 1}$, and $|\omega| = 1$.

Theorem 2 (Positive definiteness of Γ). *Under assumptions* (vi) *and* (vii), *the covariance matrix* Γ, *given in* (3.4), *is non-singular.*

Proof. The following function $f_X : \mathbb{C}\backslash\{0\} \to \mathbb{C}^{d\times d}$ is related to Γ,

$$f_X(\omega) := \sum_{k=-s}^{s} F_k \omega^k, \quad \text{for all } \omega \in \mathbb{C}\backslash\{0\}. \tag{4.4}$$

As $F_k = F_{-k}^\top$, we have that $f_X(\omega) \in \mathbb{C}_{\text{sym}}^{d\times d}$, where $\mathbb{C}_{\text{sym}}^{d\times d}$ is the space of all Hermitian-symmetric $d \times d$ matrices with complex elements.

Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$. We consider the space

$$L_d^2(\mathbb{T}) = \left\{ g : g(\omega) = \sum_{k=-\infty}^{\infty} g_k \omega^k, \text{ for } \omega \in \mathbb{T}, \; g_k \in \mathbb{C}^{d\times d}, \; \sum_{k=-\infty}^{\infty} ||g_k||_{\text{F}}^2 < \infty \right\}.$$

It is a Hilbert space with the scalar product

$$(g,h)_{L^2_d} := \frac{1}{d}\sum_{k=-\infty}^{\infty} \operatorname{tr}(g_k h_k^*), \quad \text{for all } g,h \in L^2_d(\mathbb{T}),$$

where h_k^* is the complex-adjoint matrix, $(h_k^*)_{ij} = (\bar{h}_k)_{ji}$, for $i, j = 1, \ldots, d$.

Let

$$f(\omega) := \sum_{k=-\infty}^{\infty} f_k \omega^k, \quad \text{for all } |\omega| = 1$$

be a continuous function that takes values in $\mathbb{C}^{d\times d}$. Consider a Laurent operator

$$M_f : L^2(\mathbb{T}) \to L^2(\mathbb{T}), \quad M_f : g \mapsto fg. \tag{4.5}$$

A matrix representation of M_f with respect to the sequence of functions $\mathscr{E} := \{\omega^k, \ |\omega|=1 : k \in \mathbb{Z}\}$ has the form

$$\begin{bmatrix} \ddots & \ddots & \ddots & & & \\ \cdots & f_1 & f_0 & f_{-1} & \cdots & \\ & \cdots & f_1 & f_0 & f_{-1} & \cdots \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It means that

$$M_f(g_q\omega^q) = \sum_{k=-\infty}^{\infty} f_k g_q \omega^{k+q} = \sum_{k_1=-\infty}^{\infty} f_{k_1-q} g_q \omega^{k_1},$$

therefore to find $M_f(g_q\omega^q)$, we have to perform the multiplication

$$\begin{array}{cc} & \begin{array}{ccccc} & & 0 & 1 & \end{array} \\ \begin{array}{c} \\ 0 \\ \\ q \\ \\ \end{array} & \begin{bmatrix} \ddots & \ddots & \ddots & & \\ \cdots & f_1 & f_0 & f_{-1} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & f_{q+1} & f_q & f_{q-1} & \cdots \\ & & \ddots & \ddots & \ddots \end{bmatrix} \end{array} \times \begin{array}{c} \\ \\ \\ q \\ \\ \end{array} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g_q \\ 0 \end{bmatrix} = \begin{array}{c} \\ 0 \\ 1 \\ \\ \end{array} \begin{bmatrix} \vdots \\ f_{1-q} g_q \\ f_{-q} g_q \\ f_{-1-q} g_q \\ \vdots \end{bmatrix}.$$

Let

$$\begin{aligned} H_m^2(\mathbb{T}) &= \{g \in L_d^2(\mathbb{T}) : g_k = 0, \ k \in \mathbb{Z}\setminus\{0,1,\ldots,m-1\}\} \\ &= \left\{ g : g(\omega) = \sum_{k=0}^{m-1} g_k \omega^k, \ g_k \in \mathbb{C}^{d\times d}, \ 0 \leqslant k \leqslant m-1 \right\}. \end{aligned}$$

Let P be the orthogonal projection of $L_d^2(\mathbb{T})$ onto $H_m^2(\mathbb{T})$. Then the operator

$$T_f : H_m^2(\mathbb{T}) \to H_m^2(\mathbb{T}), \quad T_f := PM_fP^*$$

has a matrix representation

$$\begin{bmatrix} f_0 & f_{-1} & \cdots & f_{m-1} \\ f_1 & f_0 & \cdots & f_{m-2} \\ \vdots & \ddots & \ddots & \vdots \\ f_{m-1} & \cdots & f_1 & f_0 \end{bmatrix}$$

with respect to the set of functions $\mathscr{E}_{m-1} := \{\omega^k,\ |\omega| = 1 : k = 0, 1, \ldots, m-1\}$. We have

$$\sigma(M_f) = \bigcup_{\omega \in \mathbb{T}} \sigma(f(\omega))$$

(Recall that $f(\omega) \in \mathbb{C}^{d\times d}$, $\omega \in \mathbb{T}$, and for all $\omega \in \mathbb{T}$, $\sigma(f(\omega))$ is a finite set containing all the eigenvalues of $f(\omega)$.)

For the function f_X given in (4.4), we have that M_{f_X} is a self-adjoint operator, therefore

$$\sigma(T_{f_X}) \subset \sigma(M_{f_X}) \subset \mathbb{R}$$

and

$$\lambda_{\max}(\Gamma) = \lambda_{\max}(T_{f_X}) \leqslant \max_{|\omega|=1} \sigma(f_X(\omega)).$$

We mention that, for all $\omega \in \mathbb{T}$, $\sum_{k=-s}^{s} F_k \omega^k$ is a (complex) positive semidefinite matrix. Indeed, for all $v \in \mathbb{C}^{d\times 1}$, we have

$$0 \leqslant \sum_{i,j=1}^{m} (\omega^{-i} v)^* F_{ij} (\omega^{-j} v) = \sum_{i,j=1}^{m} v^* F_{i-j} v \cdot \omega^{i-j} = \sum_{k=-m+1}^{m-1} (v^* F_k v) \omega^k (m - |k|).$$

(Here $v^* := [\bar{v}_1 \cdots \bar{v}_d] \in \mathbb{C}^{1\times d}$.) But $F_k = 0$ for $|k| \geqslant s$. Therefore, we have

$$\sum_{k=-s+1}^{s-1} (v^* F_k v) \omega^k \left(1 - \frac{|k|}{m}\right) \geqslant 0.$$

For m tending to infinity, we obtain

$$v^* \left(\sum_{k=-s+1}^{s-1} F_k \omega^k \right) v \geqslant 0, \quad \text{for all } v \in \mathbb{C}^{d\times 1}.$$

Therefore $f_X(\omega)$ is positive semidefinite for all $\omega \in \mathbb{T}$, and $\sigma(f_X(\omega)) \subset [0, \infty)$, for all $\omega \in \mathbb{T}$. Thus

$$\sigma(\Gamma) = \sigma(T_{f_X}) \subset \bigcup_{\omega \in \mathbb{T}} \sigma(f_X(\omega)) \subset [0, \infty). \tag{4.6}$$

Under conditions (vi) and (vii), for all $X \in \Theta_X$, the matrix Γ is non-singular. Indeed, for the right-hand side of (4.3), we have

$$||X_s v||^2 = ||(X_s^\top X_s)^{1/2} v||^2$$

and by condition (vi), $(X_s^\top X_s)^{1/2}$ is positive definite; therefore (4.3) implies that for all $X \in \Theta_X$ and $\omega \in \mathbb{T}$,

$$\lambda_{\min}\left(X_{\text{ext}}^\top \sum_{k=-s}^{s} V_k \omega^k X_{\text{ext}}\right) = \lambda_{\min}(f_X(\omega)) > 0$$

and then from (4.6), we obtain

$$\sigma(\Gamma) \subset \bigcup_{\omega \in \mathbb{T}} \sigma(f_X(\omega)) \subset (0, \infty). \qquad \square$$

5. Modification of the estimator and further assumptions

The residual $R(X) = AX - B$ is an affine function of the parameter X. It can be written (in a more homogeneous way) as $R(X) = CX_{\text{ext}}$, where $X_{\text{ext}} = \begin{bmatrix} X \\ -I \end{bmatrix}$. For arbitrary $Z \in \mathbb{R}^{(n+d)\times d}$, we can view R as a function of Z via $R(Z) = CZ$. Consequently, $r(Z) = \text{vec}(R^\top(Z))$ becomes a function of Z. The same reasoning applies to $\tilde{R}$ and $\tilde{r}$. As a result the cost function Q and the weight matrix Γ also become functions of Z,

$$Q(Z) := \text{vec}^\top((CZ)^\top)\ \Gamma(Z)^{-1}\ \text{vec}((CZ)^\top), \tag{5.1}$$

$$\Gamma(Z) := \mathbf{E}(\text{vec}((\tilde{C}Z)^\top)\text{vec}^\top((\tilde{C}Z)^\top)). \tag{5.2}$$

With some abuse of notation, we will write $Q(Z)$, $\Gamma(Z)$ and $Q(X)$, $\Gamma(X)$ at the same time. The distinction which function is meant, will be clear from the dimensions of the argument. Clearly, $Q(X_{\text{ext}}) = Q(X)$ and $\Gamma(X_{\text{ext}}) = \Gamma(X)$.

For $X \in \Theta_X$, consider

$$Z = X_{\text{ext}}(X_s^\top X_s)^{-1/2} = \begin{bmatrix} Z_{\text{fA}} \\ Z_s \\ Z_{\text{fB}} \end{bmatrix}, \tag{5.3}$$

where the blocks Z_{fA}, Z_s, and Z_{fB} have the same dimension as the corresponding blocks X_{fA}, X_s, and X_{fB}. Then in (5.3), Z_s has the property $Z_s^\top Z_s = I_d$. We introduce a parameter set for Z,

$$\Theta_Z := \text{cl}\{X_{\text{ext}}(X_s^\top X_s)^{-1/2} : X \in \Theta_X\}, \tag{5.4}$$

where "cl" denotes the closure in the corresponding space $\mathbb{R}^{(n+d)\times d}$, and mention that

$$\Theta_Z \subset \Theta := \{Z \in \mathbb{R}^{(n+d)\times d} : Z_s^\top Z_s = I_d\}.$$

Denote

$$\Theta_{Z_s} := \left\{ Z_s : Z = \begin{bmatrix} Z_{\text{fA}} \\ Z_s \\ Z_{\text{fB}} \end{bmatrix} \in \Theta_Z \right\} \tag{5.5}$$

and

$$\Theta_{\mathrm{s}} := \{Z_{\mathrm{s}} \in \mathbb{R}^{(n_{\mathrm{s}}+d_{\mathrm{s}})\times d} : Z_{\mathrm{s}}^{\top} Z_{\mathrm{s}} = I_d\}. \tag{5.6}$$

Then Θ_{s} is a compact set in $\mathbb{R}^{(n_{\mathrm{s}}+d_{\mathrm{s}})\times d}$ and $\Theta_{Z_{\mathrm{s}}}$ is a compact subset of Θ_{s}. Under condition (4.3), we have

$$\lambda_{\min}(f_Z(\omega)) \geqslant \lambda_0, \quad \text{for all } Z \in \Theta_Z, \text{ and } \omega \in \mathbb{T}, \tag{5.7}$$

where

$$f_Z(\omega) := Z^{\top} \left(\sum_{k=-s}^{s} V_k \omega^k \right) Z. \tag{5.8}$$

A regularized problem, corresponding to (3.2), is

$$\min_{Z \in \Theta_Z} Q(Z). \tag{5.9}$$

Let $\hat{Z}$ be a solution of (5.9), and $\hat{Z} = \begin{bmatrix} \hat{Z}' \\ \hat{Z}'' \end{bmatrix}$, where $\hat{Z}'' \in \mathbb{R}^{d\times d}$. Suppose that $\operatorname{rank}(\hat{Z}'') = d$. Then

$$\hat{Z} \in \{X_{\mathrm{ext}}(X_{\mathrm{s}}^{\top} X_{\mathrm{s}})^{-1/2} : X \in \Theta_X\} \tag{5.10}$$

and for certain $\hat{X} \in \Theta_X$,

$$\hat{Z} = \hat{X}_{\mathrm{ext}}(\hat{X}_{\mathrm{s}}^{\top} \hat{X}_{\mathrm{s}})^{-1/2}.$$

Now $Q(\hat{Z}) = Q(\hat{X})$, therefore $\hat{X}$ satisfies Definition 1, and it is the STLS estimator of X_0. Let $\operatorname{rank}(\hat{Z}'') = d$. Then we renew $\hat{X}$ from $\hat{Z}$ by

$$\hat{X}_{\mathrm{ext}} = \begin{bmatrix} -\hat{Z}'(\hat{Z}'')^{-1} \\ -I_d \end{bmatrix}, \quad \hat{X} = -\hat{Z}'(\hat{Z}'')^{-1}.$$

Below, under further assumptions, we will show that in probability for m tending to infinity (i.e., with probability tending to one as $m \to \infty$), there exists a solution $\hat{Z}$ of (5.9), which satisfies (5.10). Therefore the asymptotic properties of the STLS estimator will follow the asymptotic properties of $\hat{Z}$, which satisfies (5.10) and delivers a minimum to $Q(Z)$ on Θ_Z.

We list the additional assumptions.

(viii) There exists $\gamma \geqslant 2$ with $\gamma > d(n_{\mathrm{s}} + d_{\mathrm{s}} - (d+1)/2)$, such that

$$\sup_{i \geqslant 1,\ 1 \leqslant j \leqslant n+d} \mathbf{E}|\delta_{ij}|^{2\gamma} < \infty. \tag{5.11}$$

(ix) $\frac{\lambda_{\min}(A_0^{\top} A_0)}{\sqrt{m}} \to \infty$, as $m \to \infty$.

(x) $\frac{\lambda^2_{\min}(A_0^\top A_0)}{\lambda_{\max}(A_0^\top A_0)} \to \infty$, as $m \to \infty$.

Note that conditions (ix), (x), and (viii) with $\gamma = 2$ are exactly Gallo's conditions of weak consistency for $d = 1$ in a homoscedastic unstructured case (Gallo, 1982). Due to condition (ix), $A_0^\top A_0$ is non-singular for large m, and for that m, the matrix X_0 satisfying (1.2) is unique.

6. Properties of the inverse weight matrix $\mathbf{\Gamma}^{-1}$

For $Z \in \Theta_Z$, the function f_Z, defined in (5.8), depends on Z through $Z_\mathrm{s} \in \Theta_{Z_\mathrm{s}}$, and Θ_{Z_s} is a compact set in $\mathbb{R}^{(n_\mathrm{s}+d_\mathrm{s})\times d}$. Therefore due to (5.7), there exists such an $\varepsilon > 0$, that for all $Z \in \Theta_Z$ and $1-\varepsilon < |\omega| < 1+\varepsilon$, the function $f_Z(\omega)$ is non-singular, and it is analytic on the disk $1-\varepsilon < |\omega| < 1+\varepsilon$. Then the function

$$h_Z(\omega) := f_Z^{-1}(\omega) \quad \text{for all } 1-\varepsilon < |\omega| < 1+\varepsilon$$

is analytic, and it can be expanded as

$$h_Z(\omega) = \sum_{k=-\infty}^{\infty} H_k \omega^k \quad \text{for all } 1-\varepsilon < |\omega| < 1+\varepsilon,$$

where $H_k \in \mathbb{C}^{d\times d}$, $k \in \mathbb{Z}$, and the series converges pointwise. The properties $H_k \in \mathbb{R}^{d\times d}$ and $H_k = H_{-k}^\top$, $k \in \mathbb{Z}$, are inherited from the coefficients of the function f_Z.

We prove that H_k have exponential decay, i.e., that there exist constants $\mathtt{c}_1, \mathtt{c}_2 > 0$, such that for all $k \in \mathbb{Z}$ and all $Z \in \Theta_Z$,

$$||H_k||_\mathrm{F} \leqslant \mathtt{c}_1 \exp(-\mathtt{c}_2 \cdot |k|). \tag{6.1}$$

Indeed, by the Cauchy formula, we have

$$H_k = \frac{1}{2\pi\mathbf{i}} \oint_{|\omega|=1} \frac{h_Z(\omega)}{\omega^{k+1}}\, \mathrm{d}\omega, \tag{6.2}$$

where $\mathbf{i}$ denotes the imaginary unit $\sqrt{-1}$. Let $k \geqslant 0$. The function $h_Z(\omega)/\omega^{k+1}$ is analytic on the disk $1-\varepsilon < |\omega| < 1+\varepsilon$, therefore

$$H_k = \frac{1}{2\pi\mathbf{i}} \oint_{|\omega|=1+\varepsilon/2} \frac{h_Z(\omega)}{\omega^{k+1}}\, \mathrm{d}\omega. \tag{6.3}$$

But $||h_Z(\omega)||_\mathrm{F} = ||f_Z^{-1}(\omega)||_\mathrm{F} \leqslant \mathtt{c}_1$, for $|\omega| \leqslant 1+\varepsilon/2$. This is true because $f_Z(\omega)$ is non-singular for $Z \in \Theta_Z$, $|\omega| \leqslant 1+\varepsilon/2$, depends on Z through $Z_\mathrm{s} \in \Theta_Z$, which is compact, and is continuous in both Z_s and ω. Then we change the variable $\omega = (1+\varepsilon/2)\omega_1$ in (6.3), and obtain

$$\begin{aligned}
\|H_k\|_{\mathrm F} &= \left\| \frac{1}{2\pi \mathbf{i}} \int\limits_{|\omega_1|=1} \frac{h_Z((1+\varepsilon/2)\omega_1)}{\omega_1^{k+1}} \frac{\mathrm d\omega_1}{(1+\varepsilon/2)^k} \right\|_{\mathrm F} \\
&\leqslant \frac{1}{2\pi} \int\limits_{|\omega_1|=1} \frac{\|h_Z((1+\varepsilon/2)\omega_1)\|_{\mathrm F}}{|\omega_1|^{k+1}} \frac{|\mathrm d\omega_1|}{(1+\varepsilon/2)^k} \\
&\leqslant \mathtt{c}_1 \exp(-\mathtt{c}_2 k),
\end{aligned}$$

where $\mathtt{c}_2 := \ln(1+\varepsilon/2)$. Now, (6.1) follows immediately, because for $k<0$, $\|H_k\|_{\mathrm F} = \|H_{-k}^{\top}\|_{\mathrm F} = \|H_{-k}\|_{\mathrm F}$.

Next, consider a matrix representation of the operator M_{f_Z} (see (4.5)) in $L_d^2(\mathbb{T})$,

$$\begin{bmatrix}
\ddots & & \ddots & \ddots & \ddots & & \ddots & 0 & \cdots & 0 \\
0 & F_s & \cdots & F_1 & F_0 & F_{-1} & \cdots & F_{-s} & \ddots & \vdots \\
\vdots & \ddots & F_s & \cdots & F_1 & F_0 & F_{-1} & \cdots & F_{-s} & 0 \\
0 & \cdots & 0 & \ddots & & \ddots & \ddots & \ddots & & \ddots
\end{bmatrix}. \tag{6.4}$$

We have

$$(M_{f_Z})^{-1} = M_{f_Z^{-1}} = M_{h_Z},$$

therefore the inverse matrix in (6.4) is a matrix representation of the operator M_{h_Z}, namely

$$\begin{bmatrix}
\ddots & \ddots & \ddots & & & \\
\cdots & H_1 & H_0 & H_{-1} & \cdots & \\
 & \cdots & H_1 & H_0 & H_{-1} & \cdots \\
 & & & \ddots & \ddots & \ddots
\end{bmatrix}. \tag{6.5}$$

The Γ matrix (5.2) is a submatrix of (6.4). The elements of Γ^{-1} are close to the corresponding submatrix of (6.5)

$$\Phi := \begin{bmatrix}
H_0 & H_{-1} & \cdots & H_{-m+1} \\
H_1 & H_0 & \cdots & H_{-m+2} \\
\vdots & \ddots & \ddots & \vdots \\
H_{m-1} & \cdots & H_1 & H_0
\end{bmatrix} = [\Phi_{ij}]_{i=1,\ldots,m}^{j=1,\ldots,m}.$$

To find the relation between Γ^{-1} and Φ, we consider

$$\Gamma\Phi = \left[\begin{array}{ccc} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{s+1,1} & \cdots & \alpha_{s+1,m} \\ \hline \underset{(m-2s-2)d\times(s+1)d}{0} & \underset{(m-2s-2)d}{I} & \underset{(m-2s-2)d\times(s+1)d}{0} \\ \hline \alpha_{m-s,1} & \cdots & \alpha_{m-s,1} \\ \vdots & & \vdots \\ \alpha_{m,1} & \cdots & \alpha_{m,m} \end{array}\right]. \tag{6.6}$$

Here $\alpha_{ij} \in \mathbb{R}^{d\times d}$, and in the middle we have the structure $[0 \;\; I_{(m-2s-2)d} \;\; 0]$ because of the particular banded structure of Γ. Next, (6.6) can be written as

$$\Gamma\Phi = I_{md} + D, \tag{6.7}$$

where

$$D = \left[\begin{array}{ccc} \beta_{11} & \cdots & \beta_{1m} \\ \vdots & & \vdots \\ \beta_{s+1,1} & \cdots & \beta_{s+1,m} \\ \hline 0 & \cdots & 0 \\ \hline \beta_{m-s,1} & \cdots & \beta_{m-s,1} \\ \vdots & & \vdots \\ \beta_{m,1} & \cdots & \beta_{m,m} \end{array}\right]. \tag{6.8}$$

The entries $\beta_{ij} \in \mathbb{R}^{d\times d}$ are uniformly bounded for $Z \in \Theta_Z$. Now, we are looking for a sharper bound for the entries of D. Consider β_{ij} with $1 \leqslant i \leqslant s+1$, $j > s+1$. We have, see (6.7),

$$\beta_{ij} = -\alpha_{ij} = [F_{i-1} \cdots F_{-s} \;\; 0 \cdots 0] \begin{bmatrix} H_{-j+1} \\ \vdots \\ H_{m-j} \end{bmatrix} = \sum_{k=-j+1}^{-j+s+1} \tilde{F}_k H_k,$$

where $\tilde{F}_k$ are uniformly bounded matrices. Then, due to (6.1),

$$\begin{aligned} \|\beta_{ij}\|_{\mathrm{F}} &\leqslant \text{const}\, \mathtt{c}_1 \sum_{k=-j+1}^{-j+s+1} \exp(-\mathtt{c}_2 \cdot |k|) \\ &\leqslant \text{const}\, \mathtt{c}_1 (s+1) \exp(-\mathtt{c}_2\,(j-s-1)) \\ &\leqslant \text{const}\, \exp(-\mathtt{c}_2 j). \end{aligned} \tag{6.9}$$

Similarly, for β_{ij}, $m-s \leqslant i \leqslant m$, we have

$$\|\beta_{ij}\|_{\mathrm{F}} \leqslant \text{const}\, \exp(-\mathtt{c}_2(m-j)).$$

Finally, from (6.7), we have

$$\Gamma^{-1} = \Phi - \Gamma^{-1} D. \tag{6.10}$$

For the consistency proof, we use (6.10) intensively to bound the cost function Q.

Remark 3. Consistency usually requires that the noise has finite fourth-order moments. Here in (vii) higher moments are used. There are two reasons for this: (a) in the presence of structured relations, we do not demand the parameter set Θ_X to be bounded, cf. Pintelon and Schoukens (2001, Chapter 7), where Θ_X is a compact set, and (b) the problem is multivariate, i.e., d can be greater than 1, cf. Gallo (1982) where $d=1$ and fourth-order moments are required, and Kukush and Van Huffel (2004) where in a multivariate unstructured problem higher-order moments are used.

7. Main results

Denote

$$\mu_m := \frac{m^{1/4}}{\lambda_{\min}^{1/2}(A_0^\top A_0)} + \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)}. \tag{7.1}$$

We present the consistency statements.

Theorem 3. *Under conditions* (i)–(x), *the STLS estimator* $\hat{X}$ *converges in probability to the true value* X_0, *as m tends to infinity, i.e.,*

$$\hat{X} \xrightarrow{\mathrm{P}}, X_0 \quad \text{as } m \to \infty.$$

Moreover,

$$\|\hat{X} - X_0\|_{\mathrm{F}} = \mu_m \mathrm{O_p}(1).$$

Theorem 4. *Suppose that conditions* (i)–(viii) *hold. Assume additionally that for* γ *from condition* (viii), *the following series converge*

(xi) $\displaystyle\sum_{m=m_0}^{\infty} \left(\frac{\lambda_{\max}(A_0^\top A_0)}{\lambda_{\min}^2(A_0^\top A_0)} \right)^{\gamma} < \infty.$

(xii) $\sum_{m=m_0}^{\infty}\left(\frac{\sqrt{m}}{\lambda_{\min}(A_0^\top A_0)}\right)^{\gamma}<\infty.$

Then the STLS estimator converges to X_0 a.s., as m tends to infinity, i.e.,

$$\hat{X} \to X_0, \quad as\ m \to \infty\ a.s.$$

8. Decomposition of the cost function and lower bound

In preparation for the consistency proofs, we investigate the properties of Q, given in (5.1). Hereafter, we assume that the conditions of Theorem 3 hold.

Introduce the matrix $Z_0 \in \Theta_Z$, $Z_0 := X_{0\text{ext}}(X_{0\text{s}}^\top X_{0\text{s}})^{-1/2}$, where $X_{0\text{ext}}$ and $X_{0\text{s}}$ are the matrices X_{ext} and X_{s}, for $X=X_0$. Let $Z_0=\begin{bmatrix} Z_0' \\ Z_0'' \end{bmatrix}$, with $Z_0'' \in \mathbb{R}^{d\times d}$. Then $X_0=-Z_0'(Z_0'')^{-1}$. Define

$$\Delta V := Z' + X_0 Z''. \tag{8.1}$$

In Kukush and Van Huffel (2004, Section 5), it is shown that $\Delta V=0$ if and only if rank $(Z'')=d$ and $X_0=-Z_0'(Z_0'')^{-1}$. Then

$$C_0 Z = A_0 \Delta V.$$

Denote

$$r_{0i} := \mathbf{E} r_i(Z) = Z^\top c_{0i} = (\Delta V)^\top a_{0i}. \tag{8.2}$$

We have

$$\mathbf{E}Q(Z) = \sum_{i,j=1}^{m} r_{0i}^\top M_{ij} r_{0j} + \sum_{i,j=1}^{m} \mathbf{E}\tilde{r}_i^\top M_{ij}\tilde{r}_j, \tag{8.3}$$

where $M_{ij}=M_{ij}(Z)$ are $d\times d$-blocks of $\Gamma^{-1}(Z)$. But, see (5.2),

$$\sum_{i,j=1}^{m} \mathbf{E}\tilde{r}_i^\top M_{ij}\tilde{r}_j = \mathbf{E}(\tilde{r}^\top \Gamma^{-1}\tilde{r}) = \operatorname{tr}(\mathbf{E}(\Gamma^{-1}\tilde{r}\tilde{r}^\top)) = \operatorname{tr}(\Gamma^{-1}\Gamma) = md. \tag{8.4}$$

From (8.2)–(8.4), we have

$$\mathbf{E}Q(Z) - \mathbf{E}Q(Z_0) = \sum_{i,j=1}^{m} (\Delta V^\top a_{0i})^\top M_{ij} (\Delta V^\top a_{0i}). \tag{8.5}$$

Now,

$$\lambda_{\min}(\Gamma^{-1}) = \frac{1}{\lambda_{\max}(\Gamma)} \geqslant \text{const} = \frac{1}{\max_{\omega\in\mathbb{T},\ Z\in\Theta_Z}\sigma(f_Z(\omega))} > 0.$$

Therefore from (8.5), we have

$$\mathbf{E}Q(Z) - \mathbf{E}Q(Z_0) \geqslant \text{const}\ ||A_0\Delta V||_{\text{F}}^2 \geqslant \text{const}\ \lambda_{\min}(A_0^\top A_0)\ ||\Delta V||_{\text{F}}^2. \tag{8.6}$$

Next,

$$\tilde{Q}(Z) := Q(Z) - \mathbf{E}Q(Z) = U_1 + 2U_2, \tag{8.7}$$

where

$$U_1 := \sum_{i,j=1}^{m} (\tilde{r}_i^\top M_{ij}\tilde{r}_j - \mathbf{E}\,\tilde{r}_i^\top M_{ij}\tilde{r}_j),$$
$$U_2 := \sum_{i,j=1}^{m} r_{0i}^\top M_{ij}\tilde{r}_j = \sum_{i,j=1}^{m} a_{0i}^\top \Delta V M_{ij}\tilde{r}_j.$$

The summands U_1 and U_2 are bounded in Appendix A. The derivation of the bounds is based on the properties of the inverse weight matrix Γ^{-1} and the results on stochastic fields and moment inequalities, collected in Appendix B. From (8.7), (8.6), (A.19), (A.16) and taking into account that $U_2(Z_0)=0$, we obtain, for $Z \in \Theta_Z$ and for some positive constant c,

$$Q(Z) - Q(Z_0) \geqslant \mathsf{c} \cdot \lambda_{\min}(A_0^\top A_0)||\Delta V||_{\mathrm{F}}^2 + \sqrt{m}\xi_m(Z) + \sqrt{\lambda_{\max}(A_0^\top A_0)}||\Delta V||_{\mathrm{F}} \cdot \rho_m(Z), \tag{8.8}$$

where

$$\mathbf{P}\{\sup_{Z\in\Theta_Z} |\xi_m(Z)| > a\} \leqslant \text{ const}\, a^{-\gamma}, \quad \text{for } a > 0 \tag{8.9}$$

and

$$\mathbf{P}\{\sup_{Z\in\Theta_Z} |\rho_m(Z)| > a\} \leqslant \text{const}\, a^{-2\gamma}, \quad \text{for } a > 0. \tag{8.10}$$

9. Proofs of the main results

First ,we prove that the regularized problem (5.9) has a solution, with probability tending to one, as the sample size m grows to infinity.

9.1. *Existence of solution of* $\min_{Z\in\Theta_Z} Q(Z)$

We suppose that conditions (i)–(x) hold. We start with the function

$$q(Z) := ||\Delta V||_{\mathrm{F}}^2 = ||Z' + X_0 Z''||_{\mathrm{F}}^2, \quad \text{where } Z = \begin{bmatrix} Z' \\ Z'' \end{bmatrix} \in \Theta_Z. \tag{9.1}$$

Below, for a matrix $M \in \mathbb{R}^{p\times q}$, we use an operator norm,

$$||M|| := \sup_{u\in\mathbb{R}^q} \frac{||Mu||_{\mathbb{R}^p}}{||u||_{\mathbb{R}^q}}. \tag{9.2}$$

We mention that for $Z_{\mathrm{s}} \in \Theta_{Z_{\mathrm{s}}}$, we have $\|Z_{\mathrm{s}}\|=1$, because the columns of Z_{s} are orthogonal unit vectors. Denote

$$\mathsf{c}_0 := 2\|X_{0\mathrm{ext}}(X_{0\mathrm{s}}^\top X_{0\mathrm{s}})^{-1}X_{0\mathrm{s}}^\top\|, \tag{9.3}$$

where the operator norm (9.2) is used. The constant c_0 is positive, see Kukush and Van Huffel (2004, Section 7.1).

Now, we show that

$$\inf_{Z\in\Theta_Z,\ \|Z\|\geqslant \mathsf{c}_0} q(Z) \geqslant \nu^2, \quad \text{for certain fixed } \nu>0. \tag{9.4}$$

Indeed, suppose that for a certain sequence $\{Z(l),\ l\geqslant 1\}\subset\Theta_Z$, with $\|Z(l)\|\geqslant \mathsf{c}_0$, we have $\lim_{l\to\infty} q(Z(l))=0$. Then (9.1) implies $Z(l)+X_{0\mathrm{ext}}Z''(l)\to 0$, and $Z_{\mathrm{s}}(l)+X_{0\mathrm{s}}Z''(l)\to 0$, as $l\to\infty$. Using assumption (vi), we have

$$\lim_{l\to\infty}(Z''(l)+(X_{0\mathrm{s}}^\top X_{0\mathrm{s}})^{-1}X_{0\mathrm{s}}^\top Z_{\mathrm{s}}(l))=0$$

and

$$\lim_{l\to\infty}(Z(l)-X_{0\mathrm{ext}}(X_{0\mathrm{s}}^\top X_{0\mathrm{s}})^{-1}X_{0\mathrm{s}}^\top Z_{\mathrm{s}}(l))=0.$$

But this does not hold, because $\|Z(l)\|\geqslant \mathsf{c}_0$, and

$$\|X_{0\mathrm{ext}}(X_{0\mathrm{s}}^\top X_{0\mathrm{s}})^{-1}X_{0\mathrm{s}}^\top Z_{\mathrm{s}}(l)\|\leqslant \|X_{0\mathrm{ext}}(X_{0\mathrm{s}}^\top X_{0\mathrm{s}})^{-1}X_{0\mathrm{s}}^\top\|\cdot\|Z_{\mathrm{s}}(l)\|=\mathsf{c}_0/2.$$

We got a contradiction and this proves (9.4).

Now, we rewrite (8.8) in the form

$$\begin{aligned}&\frac{1}{\lambda_{\min}(A_0^\top A_0)}(Q(Z)-Q(Z_0))\\ &\quad\geqslant \mathsf{c}\cdot\left(\|\Delta V\|_{\mathrm{F}}+\frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)}\tilde\rho_m(Z)\right)^2\\ &\qquad+\frac{\sqrt{m}}{\lambda_{\min}(A_0^\top A_0)}\xi_m(Z)-\mathsf{c}\,\frac{\lambda_{\max}(A_0^\top A_0)}{\lambda_{\min}^2(A_0^\top A_0)}\tilde\rho_m(Z).\end{aligned} \tag{9.5}$$

Here $\tilde\rho_m(Z)$ satisfies (8.10). Introduce a random variable

$$\mathscr{A}_m := \inf_{Z\in\Theta_Z,\ \|Z\|\geqslant \mathsf{c}_0}(Q(Z)-Q(Z_0)).$$

We show that

$$\mathbf{P}\{\mathscr{A}_m<0\}\to 0 \quad \text{as } m\to\infty. \tag{9.6}$$

Suppose that $\mathscr{A}_m < 0$ holds. Then there exists $Z^* \in \Theta_Z$, such that $||Z^*|| \geqslant \mathsf{c}_0$ and $Q(Z^*) - Q(Z_0) < 0$. Let $\Delta V^* := \Delta V|_{Z=Z^*}$. From (9.5), we have

$$\left(||\Delta V^*||_{\mathrm{F}} + \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \tilde{\rho}_m(Z^*) \right)^2 < \frac{\sqrt{m}}{\lambda_{\min}(A_0^\top A_0)} \frac{\sup_{Z\in\Theta_Z} |\xi_m(Z)|}{\mathsf{c}} + \frac{\lambda_{\max}(A_0^\top A_0)}{\lambda_{\min}^2(A_0^\top A_0)} \left(\sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \right)^2 := \Lambda_m.$$

Due to (x), in probability for m tending to infinity,

$$\frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \leqslant v,$$

where v comes from (9.4), then we obtain, due to (9.4),

$$\left(v - \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \right)^2 < \Lambda_m.$$

Therefore, we have

$$\mathbf{P}\{\mathscr{A}_m < 0\} \leqslant \mathbf{P}\left\{ \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \geqslant v \right\} + \mathbf{P}\left\{ \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \leqslant v, \text{ and } \left(v - \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \right)^2 < \Lambda_m \right\}.$$

Using (8.10) for $\tilde{\rho}_m(Z)$, we have

$$\mathbf{P}\{\mathscr{A}_m < 0\} \leqslant \mathrm{const} \left(\frac{\lambda_{\max}(A_0^\top A_0)}{\lambda_{\min}^2(A_0^\top A_0)} \right)^\gamma + \mathbf{P}\left\{ v < 2 \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| + \frac{\sqrt[4]{m}}{\lambda_{\min}^{1/2}(A_0^\top A_0)} \frac{(\sup_{Z\in\Theta_Z} |\xi_m(Z)|)^{1/2}}{\sqrt{c}} \right\}$$
$$\leqslant \mathrm{const} \left(\left(\frac{\lambda_{\max}(A_0^\top A_0)}{\lambda_{\min}^2(A_0^\top A_0)} \right)^\gamma + \left(\frac{\sqrt{m}}{\lambda_{\min}(A_0^\top A_0)} \right)^\gamma \right).$$

And for μ_m, defined in (7.1),

$$\mathbf{P}\{\mathscr{A}_m < 0\} \leqslant \mathrm{const}(\mu_m)^{2\gamma}. \tag{9.7}$$

By conditions (ix) and (x), this implies (9.6). Then, in probability for m tending to infinity,

$$\inf_{Z\in\Theta_Z} Q(Z) = \inf_{Z\in\Theta_Z,\ ||Z|| \leqslant \mathsf{c}_0} Q(Z),$$

and the last lower bound is attained, because $Q(Z)$ is continuous on the compact set $\{Z \in \Theta_Z : ||Z|| \leqslant \mathsf{c}_0\}$. Thus the solution of (5.9) exists with probability tending to one. Moreover,

$$\mathbf{P}\{\hat{Z} = \infty \text{ or } ||\hat{Z}|| \geqslant \mathsf{c}_0\} \leqslant \mathbf{P}\{\mathscr{A}_m < 0\} \leqslant \text{const}(\mu_m)^{2\gamma}. \tag{9.8}$$

Here $\hat{Z}$ is a solution of (5.9), and $\hat{Z} = \infty$ if in (5.9), a minimum is not attained.

9.2. Proof of Theorem 3

The results of Section 9.1 imply that there exists a solution of problem (5.9) with probability tending to one, as m tends to infinity. Now, introduce the event

$$\mathscr{B}_m := \{||\hat{Z}|| \leqslant \mathsf{c}_0\},$$

with c_0 defined in (9.3). We mention that due to (9.8)

$$\mathbf{P}\{\overline{\mathscr{B}_m}\} \leqslant \mathbf{P}\{\mathscr{A}_m < 0\} \quad \text{and} \quad \mathbf{P}\{\mathscr{B}_m\} \to 0 \quad \text{as } m \to \infty. \tag{9.9}$$

Assume that $\mathscr{B}_m$ holds. From (8.8) and the definition of $\hat{Z}$, we have $Q(\hat{Z}) - Q(Z_0) \leqslant 0$, and for $\Delta\hat{V} := \hat{Z} - Z_0$,

$$\begin{aligned} ||\Delta\hat{V}||_{\mathrm{F}}^2 \leqslant \text{const} \cdot \frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| \cdot ||\Delta\hat{V}||_{\mathrm{F}} \\ + \text{const} \frac{\sqrt{m}}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\xi}_m(Z)|. \end{aligned}$$

This implies that

$$||\Delta\hat{V}||_{\mathrm{F}} \leqslant \text{const} \cdot \left(\frac{\lambda_{\max}^{1/2}(A_0^\top A_0)}{\lambda_{\min}(A_0^\top A_0)} \sup_{Z\in\Theta_Z} |\tilde{\rho}_m(Z)| + \frac{\sqrt[4]{m}}{\lambda_{\min}^{1/2}(A_0^\top A_0)} \left(\sup_{Z\in\Theta_Z} |\tilde{\xi}_m(Z)| \right)^{1/2} \right).$$

Therefore, see (8.9) and (8.10),

$$\begin{aligned} \mathbf{P}\{||\Delta\hat{V}||_{\mathrm{F}} > \delta\} \leqslant \mathbf{P}\{\overline{\mathscr{B}_m}\} + \text{const}\, \frac{1}{\delta^{2\gamma}} \left(\left(\frac{\lambda_{\max}(A_0^\top A_0)}{\lambda_{\min}^2(A_0^\top A_0)} \right)^{\gamma} + \left(\frac{\sqrt{m}}{\lambda_{\min}(A_0^\top A_0)} \right)^{\gamma} \right) \\ \leqslant \text{const}(\mu_m)^{2\gamma}(1 + 1/\delta^{2\gamma}) \quad \text{for } \delta > 0. \end{aligned} \tag{9.10}$$

In the last inequality, we used (9.9) and (9.8). Now, (9.10) implies that

$$\Delta\hat{V} \xrightarrow{\mathrm{P}} 0 \quad \text{as } m \to \infty.$$

From (8.1), we get

$$\hat{Z}' + X_0\hat{Z}'' \xrightarrow{\mathrm{P}} 0 \quad \text{as } m \to \infty.$$

To complete the proof of the convergence $\hat{X} \xrightarrow{\mathrm{P}} X_0$, as $m \to \infty$, we have to show that in probability for m tending to infinity,

$$\operatorname{rank}(\hat{Z}'') = d \quad \text{and} \quad (\hat{Z}'')^{-1} = \mathrm{O_p}(1). \tag{9.11}$$

Indeed once we show this, then $\operatorname{rank}(\hat{Z}'') = d$, implies $\hat{Z} \in \{X_{\text{ext}}(X_{\text{s}}^\top X_{\text{s}})^{-1/2} : X \in \Theta_X\}$, and

$$\hat{X} = -\hat{Z}'(\hat{Z}'')^{-1} = (X_0\hat{Z}'' + o_{\text{p}}(1))(\hat{Z}'')^{-1} = X_0 + o_{\text{p}}(1)$$

and Theorem 3 is proved. Thus we have to prove (9.11).

Consider function (9.1) on the set

$$\Theta_1 := \{Z \in \Theta_Z : ||Z|| \leqslant \mathsf{c}_0\}.$$

If $q(Z)=0$, then $Z=\begin{bmatrix} -X_0 \\ I_d \end{bmatrix} Z''$, $Z_{\text{s}}=-X_{0\text{s}}Z''$, $(Z'')^{-1}=-Z_{\text{s}}^\top X_{0\text{s}}$, and $||(\hat{Z}'')^{-1}||_{\text{F}} \leqslant \mathsf{c}_1=$ const. Hereafter, we write $||(\hat{Z}'')^{-1}||_{\text{F}} = \infty$ if Z'' is singular.

The set Θ_1 is compact and $q(Z)$ is continuous, then

$$\inf_{\substack{Z\in\Theta_1 \\ 2\mathsf{c}_1 \leqslant ||(\hat{Z}'')^{-1}||_{\text{F}} \leqslant \infty}} ||\Delta\hat{V}||_{\text{F}}^2 \geqslant v_0^2 > 0.$$

We have

$$\mathbf{P}\{2\mathsf{c}_1 \leqslant ||(\hat{Z}'')^{-1}||_{\text{F}} \leqslant \infty,\ \text{and } \mathscr{B}_m \text{ happens}\} \leqslant \mathbf{P}\{||\Delta\hat{V}||_{\text{F}}^2 \geqslant v_0^2\}, \tag{9.12}$$

and this tends to zero. Therefore (9.11) is shown and the convergence $\hat{X} \xrightarrow{\text{P}} X_0$, as $m \to \infty$, is proved.

Now, we show the second statement of Theorem 3. Bounds (9.12) and (9.10) imply that

$$\mathbf{P}\{2\mathsf{c}_1 \leqslant ||(\hat{Z}'')^{-1}||_{\text{F}} \leqslant \infty,\ \text{and } \mathscr{B}_m \text{ happens}\} \leqslant \text{const}(\mu_m)^{2\gamma}. \tag{9.13}$$

Because of

$$\hat{X} - X_0 = \Delta\hat{V}(\hat{Z}'')^{-1},$$

we have

$$\begin{aligned}\mathbf{P}\{||\hat{X} - X_0||_{\text{F}} > \delta\} = \mathbf{P}\{&\hat{Z}'' \text{ is singular or } ||\Delta\hat{V}(\hat{Z}'')^{-1}||_{\text{F}} > \delta\} \\ &\leqslant \mathbf{P}\{\mathscr{A}_m < 0\} + \mathbf{P}\{2\mathsf{c}_1 \leqslant ||(\hat{Z}'')^{-1}||_{\text{F}} \leqslant \infty,\ \text{and } \mathscr{B}_m \text{ happens}\} \\ &\quad + \mathbf{P}\{||\hat{V}||_{\text{F}} > \delta/(2\mathsf{c}_1)\}\end{aligned}$$

and hence with (9.10), (9.13), and (9.7),

$$\mathbf{P}\{||\hat{X} - X_0||_{\text{F}} > \delta\} \leqslant \text{const}(1 + 1/\delta^{2\gamma})(\mu_m)^{2\gamma}. \tag{9.14}$$

then

$$\mathbf{P}\left\{\frac{||\hat{X} - X_0||_{\text{F}}}{\mu_m} > \delta\right\} \leqslant \text{const}\left((\mu_m)^{2\gamma} + \frac{1}{\delta^{2\gamma}}\right)$$

and the second statement of Theorem 3 follows. □

If the error-free columns in A and B are absent, the proof of Theorem 3 is simpler. In that case $\Theta = \Theta_{\mathrm{s}}$ is compact, and $\Theta_Z = \Theta_{Z_{\mathrm{s}}} \subset \Theta_{\mathrm{s}}$, Θ_Z is compact as well. As a result of this, part in Section 9.1 is not needed any more for the proof of Theorem 3.

9.3. Proof of Theorem 4

From (9.14) and conditions (xi), (xii), we have for such m_0, that $\lambda_{\min}(A_0^\top A_0) > 0, m \geqslant m_0$,

$$\sum_{m=m_0}^{\infty} \mathbf{P}\{||\hat{X} - X_0||_{\mathrm{F}} > \delta\} < \infty.$$

By the Borel–Cantelli lemma, this implies that

$$||\hat{X} - X_0||_{\mathrm{F}} \to 0, \quad \text{as } m \to \infty, \text{ a.s.}$$

10. Algorithm

Based on Theorem 1 and the cost function Q, given in (3.5), we describe a numerical scheme to compute the STLS estimator. We have to minimize the function Q on Θ_X. Suppose that X_0 is an interior point of Θ_X. Then by Theorem 3 with probability tending to one, $\hat{X}$ is a root of the equation $\mathrm{d}Q(X)/\mathrm{d}X = 0$.

We have for $H \in \mathbb{R}^{n\times d}$,

$$\frac{1}{2}Q'(X)H = \begin{bmatrix} r_1(X) \\ \vdots \\ r_m(X) \end{bmatrix}^\top \Gamma^{-1} \begin{bmatrix} H^\top a_1 \\ \vdots \\ H^\top a_m \end{bmatrix} - \frac{1}{2}\begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}^\top \Gamma^{-1}\left(\frac{\mathrm{d}\Gamma}{\mathrm{d}X}H\right)\Gamma^{-1}\begin{bmatrix} r_1 \\ \vdots \\ r_m \end{bmatrix}.$$

Now,

$$\Gamma = \mathbf{E}\begin{bmatrix} X^\top \tilde{a}_1 - \tilde{b}_1 \\ \vdots \\ X^\top \tilde{a}_m - \tilde{b}_m \end{bmatrix}[\tilde{a}_1^\top X - \tilde{b}_1^\top \ \cdots \ \tilde{a}_m^\top X - \tilde{b}_m^\top]$$

and

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}X}H = \mathbf{E}\begin{bmatrix} H^\top \tilde{a}_1 \\ \vdots \\ H^\top \tilde{a}_m \end{bmatrix}[\tilde{a}_1^\top X - \tilde{b}_1^\top \ \cdots \ \tilde{a}_m^\top X - \tilde{b}_m^\top] + \mathbf{E}\begin{bmatrix} X^\top \tilde{a}_1 - \tilde{b}_1 \\ \vdots \\ X^\top \tilde{a}_m - \tilde{b}_m \end{bmatrix}[\tilde{a}_1^\top H \cdots \tilde{a}_m^\top H].$$

Then

$$\begin{aligned}
&\frac{1}{2}Q'(X)H \\
&\quad = \operatorname{tr}\left(\sum_{i,j=1}^{m} r_i^\top M_{ij} H^\top a_j - \sum_{i,j,k,l=1}^{m} r_i^\top M_{ij} H^\top \mathbf{E}\tilde{a}_j[\tilde{a}_k^\top\ \tilde{b}_k^\top]\begin{bmatrix} X \\ -I_d \end{bmatrix} M_{kl} r_l\right) \\
&\quad = \operatorname{tr}\left(\sum_{i,j=1}^{m} a_j r_i^\top M_{ij} H^\top - \sum_{i,j,k,l=1}^{m} V_{jk}\begin{bmatrix} X \\ -I_d \end{bmatrix} M_{kl} r_l r_i^\top M_{ij} H^\top\right),
\end{aligned}$$

where $V_{jk} := \mathbf{E}[\tilde{a}_j\tilde{a}_k^\top\ \tilde{a}_j\tilde{b}_k^\top]$. Denote also

$$N_{kj} := \sum_{l=1}^{m} M_{kl} r_l \sum_{i=1}^{m} r_i^\top M_{ij} = (\Gamma^{-1} r r^\top \Gamma^{-1})_{kj}.$$

Then for $X = \tilde{X}$,

$$\frac{1}{2}\frac{\mathrm{d}Q(X)}{\mathrm{d}X} =: \Psi(X, \tilde{X}) = \sum_{i,j=1}^{m} a_j (a_i^\top X - b_i^\top) M_{ij}(\tilde{X}) - \sum_{j,k=1}^{m} V_{jk}\begin{bmatrix} X \\ -I_d \end{bmatrix} N_{kj}(\tilde{X}).$$

We have to solve the equation $\Psi(X, X)=0$. The proposed iterative algorithm is as follows:

(1) Compute an approximation $X^{(0)}$, using, e.g., the TLS estimator.
(2) Given an approximation $X^{(k)}$, we find $X^{(k+1)}$ from the linear equation $\Psi(X, X^{(k)}) = 0$.

On each step, we have to check whether $X^{(k)}$ is in the set Θ_X. If for certain k, $X^{(k)}$ is not in the set Θ_X, then we change it to $\tilde{X}^{(k)}$ which is the point in Θ_X, nearest to $X^{(k)}$.

In Markovsky et al. (2004) we implemented the proposed algorithm and compared it with the algorithms based on the structured total least norm approach (Lemmerling, 1999; Mastronardi, 2001). The comparison shows that in term of computational efficiency our proposal is competitive with the fast algorithms of Mastronardi (2001). We mention that the procedure proposed here is similar to the one for the EW-TLS problem, proposed initially in (Premoli and Rastello, 2002; Kukush and Van Huffel, 2004) and developed later on in Markovsky et al. (2002a). In Kukush et al. (2002) it is proven that the proposed algorithm for the EW-TLS problem is a contraction. For the STLS problem, this has not been shown yet.

11. Examples

11.1. Scalar model with Hankel structure

Let $n = d = 1$, $n_{\mathrm{f}} = 0$, $n_{\mathrm{s}} = 1$. The model is $ax \approx b$, where $a \in \mathbb{R}^{m\times 1}$, $x \in \mathbb{R}$, and $b \in \mathbb{R}^{m\times 1}$.

We observe the data $a_i = a_{0i} + \tilde{a}_i$, $b_i = b_{0i} + \tilde{b}_i$, with $a_{0i}x_0 = b_{0i}$, $i = 1, \ldots, m$, and want to estimate $x_0 \in \mathbb{R}$. Impose the Hankel structure

$$[a\ b] = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \\ \vdots & \vdots \\ p_m & p_{m+1} \end{bmatrix}.$$

Here the number of structural parameters is $n_p = m + 1$ and is greater than $md = m$, so that we have enough parameters in the structure, compare with the demand $n_p \geqslant md$ from assumption (ii).

We suppose that $\{\tilde{p}_i,\ i=1, 2, \ldots, m+1\}$ are independent, with zero mean and $\mathrm{var}(\tilde{p}_i) = \sigma^2_{\tilde{p}} > 0$, and $\sigma^2_{\tilde{p}}$ does not depend on i. The true values of p_i are p_{0i}, for $i = 1, 2, \ldots, n_p$.

We state a version of Theorem 4 for this scalar case.

Theorem 5. *Let the following conditions hold*:

(a) *The true value* $x_0 \in (-\infty, -1-\varepsilon] \cup [1+\varepsilon, \infty) := \Theta_x$, *and* $\varepsilon > 0$ *is known*.
(b) $\sup_{i \geqslant 1} \mathbf{E}\, \tilde{p}_i^4 < \infty$.
(c) $p_{01} \neq 0$.

Then the STLS estimator $\hat{x}$ *constructed for the parameter set* Θ_x *converges to* x_0 *a.s., as m grows to infinity, i.e.,*

$$\hat{x} \to x_0, \quad as\ m \to \infty,\ a.s.$$

We explain why condition (vii) holds now. We have $\tilde{r}_i(x) = \tilde{p}_i x - \tilde{p}_{i+1}$, $i = 1, \ldots, m$, and

$$\Gamma = \sigma^2_{\tilde{p}} \cdot \begin{bmatrix} x^2+1 & -x & & 0 \\ -x & \ddots & \ddots & \\ & \ddots & \ddots & -x \\ 0 & & -x & x^2+1 \end{bmatrix} \in \mathbb{R}^{m \times m}.$$

Then

$$f_x(\omega) = -x\omega^{-1} + (x^2+1) - x\omega, \quad \text{for } \omega \in \mathbb{C}\backslash\{0\}$$

For $|\omega| = 1$, we have,

$$f_x(\omega) = (x - \cos\varphi)^2 + \sin^2\varphi \quad \text{where } \varphi := \arg\omega.$$

In our case assumption (vii) takes the form

$$\Upsilon(x, \varphi) := \frac{(x - \cos\varphi)^2 + \sin^2\varphi}{x^2+1} \geqslant \lambda_0 > 0 \quad \text{for all } |x| \geqslant 1 + \varepsilon,\quad \varphi \in [0, 2\pi].$$

But this holds because

$$\Upsilon(x, \varphi) \to 1, \quad \text{as } x \to \infty,$$

uniformly in $\varphi \in [0, 2\pi]$, and $\Upsilon(x, \varphi) > 0$ on each set $1 + \varepsilon \leqslant |x| \leqslant L$, $\varphi \in [0, 2\pi]$. We have, $\Upsilon(x, \varphi) = 0$ if and only if $\varphi = 0$ or 2π and $x = 1$, or $\varphi = \pi$ and $x = -1$, but Θ_x is separated from ± 1, see condition (a) of Theorem 5.

Now we explain why assumption (xi) and (xii) hold, for $r = 2$. Indeed, in our model

$$p_{0,i+1} = p_{0,i} x_0 \quad \text{for } i \geqslant 1, \quad \text{so that,} \quad p_{0,i} = p_{0,1} x_0^{i-1} \quad \text{for } i \geqslant 1.$$

Then $a_0^\top a_0 = \sum_{i=1}^m (p_{0,i})^2$ grows exponentially for $|x_0| > 1$, and both $\lambda_{\min}(a_0^\top a_0) = \lambda_{\max}(a_0^\top a_0)$ grow exponentially. Therefore conditions (xi) and (xii) hold with $\gamma = 2$.

Defining Θ_x in (a), we excluded the interval $|x| < 1$. Indeed, in the case $|x_0| < 1$, $a_0^\top a_0 \leqslant (p_{0,1})^2/(1 - x_0^2)$, and condition (ix) does not hold, because $\lambda_{\min}(a_0^\top a_0)/\sqrt{m} \to 0$, as $m \to \infty$.

11.2. A model with two variables and Hankel structure

Let $n = 2$, $d = 1$, $n_\mathrm{f} = 0$, $n_\mathrm{s} = 1$. The model is

$$\begin{bmatrix} a_{11} & a_{12} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

We observe

$$a_{ij} = a_{0,ij} + \tilde{a}_{ij}, \quad b_i = b_{0,i} + \tilde{b}_i, \quad \text{for } i = 1, \ldots, m, \;\; j = 1, 2,$$

with

$$a_{0,i1} x_{0,1} + a_{0,i2} x_{0,2} = b_{0,i}, \quad \text{for } i = 1, \ldots, m$$

and we want to estimate $x_0 := \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} \in \mathbb{R}^{2\times 1}$.

Impose the Hankel structure

$$[A \;\, b] = \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_4 \\ \vdots & \vdots & \vdots \\ p_m & p_{m+1} & p_{m+2} \end{bmatrix}$$

and suppose that $\{\tilde{p}_i,\ i = 1, 2, \ldots, m+2\}$ are independent with zero mean and $\operatorname{var}(\tilde{p}_i) = \sigma_{\tilde{p}}^2 > 0$, and $\sigma_{\tilde{p}}^2$ does not depend on i and is unknown.

In this case,

$$\tilde{r}_i(x) = \tilde{p}_i x_1 + \tilde{p}_{i+1} x_2 - \tilde{p}_{i+2} \quad \text{for all } i = 1, \ldots, m,$$

and

$$f_x(\omega) = -\omega^{-2} + (x_1 x_2 - x_2)\omega^{-1} + (||x||^2 + 1) + (x_1 x_2 - x_2)\omega - \omega^2 \quad \text{for all } \omega \in \mathbb{C}\backslash\{0\}.$$

For $|\omega|=1$, we have with $\varphi := \arg\ \omega$,

$$f_x(\omega) = ||x||^2 + 1 + 2(x_1x_2 - x_2)\cos\ \varphi - 2\cos\ 2\varphi.$$

Theorem 6. *Let the following conditions hold*:

(a) *The true value* $x_0 := \begin{bmatrix} x_{0,1} \\ x_{0,2} \end{bmatrix} \in \Theta_x \subset \mathbb{R}^{2\times 1}$, *where* Θ_x *is a known closed set.*

(b) $$\inf_{x\in\Theta_x,\ \varphi\in[0,2\pi]} \frac{||x||^2 + 1 + 2(x_1x_2 - x_2)\cos\ \varphi - 2\ \cos\ 2\varphi}{||x||^2+1} > 0.$$

(c) *There exists* $\gamma > 2$, *such that* $\sup_{i\geqslant 1} \mathbf{E}|\tilde{p}_i|^{2\gamma} < \infty$.

(d) $$\frac{\lambda_{\max}(A_0^\top A_0)}{\sqrt{m}} \to \infty, \quad as\ m\to\infty.$$

(e) $$\frac{\lambda^2_{\min}(A_0^\top A_0)}{\lambda_{\max}(A_0^\top A_0)} \to \infty, \quad as\ m\to\infty.$$

Then the STLS estimator $\hat{x}$ *converges in probability to the true value* x_0 *of the parameter, as m grows to infinity, i.e.*,

$$\hat{x} \xrightarrow{\mathrm{P}} x_0, \quad as\ m\to\infty.$$

We mention that condition (b) can be written in a more explicit form, because the numerator is a quadratic function of $\cos\ \varphi$.

11.3. A model with A structured and B unstructured

As another particular case, we consider model (1.2) and in assumption (i), we demand that $A=[A_{\mathrm{f}}\ \ A_{\mathrm{s}}]$, $\tilde{A}_{\mathrm{f}}=0$, and $\tilde{A}_{\mathrm{s}}$ is independent of $\tilde{B}$. In assumption (ii), we suppose that $S:\mathbb{R}^{n_p}\to\mathbb{R}^{m\times n_{\mathrm{s}}}$, is such that $A_{\mathrm{s}}=S(p)$ for some $p\in\mathbb{R}^{n_p}$, and $V_{\tilde{p}}$ is positive definite; moreover, for $\tilde{B}$, we suppose that $\{\tilde{b}_i,\ \ i=1,2,\ldots,m\}$ are independent, and $\mathbf{E}\tilde{b}_i\tilde{b}_i^\top := V_{\tilde{b}}$ does not depend on i, and $V_{\tilde{b}}$ is positive definite.

Then the total number of structural parameters in $[A_{\mathrm{s}}\ \ B]$ equals $n_p+md\geqslant md$, and the requirement from assumption (ii) holds.

Now, we specify condition (vii). We have for V_{i-j}, defined in (4.1),

$$V_{i-j} = \mathbf{E}\begin{bmatrix} 0 \\ \tilde{a}_{\mathrm{s}i} \\ \tilde{b}_i \end{bmatrix}[0\ \tilde{a}_{\mathrm{s}j}^\top\ \ \tilde{b}_j^\top] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{E}\tilde{a}_{\mathrm{s}i}\tilde{a}_{\mathrm{s}j}^\top & 0 \\ 0 & 0 & \mathbf{E}\tilde{b}_i\tilde{b}_j^\top \end{bmatrix} \in \mathbb{R}^{(n+d)\times(n+d)}.$$

Thus for $V_{\tilde{a},i-j} := \mathbf{E}\tilde{a}_{\mathrm{s}i}\tilde{a}_{\mathrm{s}j}^\top$, we have

$$V_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_{\tilde{a},0} & 0 \\ 0 & 0 & V_{\tilde{b}} \end{bmatrix}, \qquad V_k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & V_{\tilde{a},k} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for all } k\in\mathbb{Z},\ \ k\neq 0.$$

Remember that due to condition (iv), $V_k=0$, for $|k|\geqslant n_{\mathrm{s}}$.

Denote $X_{\mathrm{ext}} = [X_{\mathrm{f}}^{\top} \quad X_{\mathrm{A}}^{\top} \quad -I_d]^{\top}$. Then

$$X_{\mathrm{ext}}^{\top}\left(\sum_{k=-n_{\mathrm{s}}+1}^{n_{\mathrm{s}}+1} V_k\omega^k\right)X_{\mathrm{ext}} = X_{\mathrm{A}}^{\top}\left(\sum_{k=-n_{\mathrm{s}}+1}^{n_{\mathrm{s}}+1} V_{\tilde{a},k}\omega^k\right)X_{\mathrm{A}} + V_{\tilde{b}}.$$

Now $X_{\mathrm{s}} = [X_{\mathrm{A}}^{\top} \quad -I_d]^{\top}$, and condition (4.3) takes the form

$$v^{\top}X_{\mathrm{A}}^{\top}\left(\sum_{k=-n_{\mathrm{s}}+1}^{n_{\mathrm{s}}+1} V_{\tilde{a},k}\omega^k\right)X_{\mathrm{A}}v + v^{\top}V_{\tilde{b}}v \geqslant \lambda_0(||X_{\mathrm{A}}v||^2 + ||v||^2) \tag{11.1}$$

for each $X \in \Theta_X$, $v \in \mathbb{R}^{d\times 1}$, and $\omega \in \mathbb{T}$; here $\lambda_0 > 0$ is fixed. One of the following two conditions is sufficient for (11.1):

(A) There exists constant a $L > 0$, such that $||X_{\mathrm{A}}||_{\mathrm{F}} \leqslant L$, for each $X \in \Theta_X$.
(B) There exists constant a $\lambda_1 > 0$, such that

$$v^{\top}X_{\mathrm{A}}^{\top}\left(\sum_{k=-n_{\mathrm{s}}+1}^{n_{\mathrm{s}}+1} V_{\tilde{a},k}\omega^k\right)X_{\mathrm{A}}v \geqslant \lambda_1||X_{\mathrm{A}}v||^2$$

for each $X \in \Theta_X$, $v \in \mathbb{R}^{d\times 1}$ and $\omega \in \mathbb{T}$.

Indeed, assume that (A) holds. Let W_{l} be the left-hand side of (11.1), and W_{r} be its right-hand side. We have

$$W_{\mathrm{l}} \geqslant \lambda_{\min}(V_{\tilde{b}})\ ||v||^2 \quad \text{and} \quad W_{\mathrm{r}} \leqslant \lambda_0\ ||v||^2(1+L^2).$$

For $\lambda_0 := \lambda_{\min}(V_{\tilde{b}})/(1+L^2) > 0$, inequality (11.1) holds.

Now, suppose that condition (B) holds. We consider two cases for fixed $\gamma > 0$

(a) $||X_{\mathrm{A}}v|| \geqslant \gamma||v||$, and
(b) $||X_{\mathrm{A}}v|| \leqslant \gamma||v||$.

In case (a), we have $W_{\mathrm{l}} \geqslant \lambda_1||X_{\mathrm{A}}v||^2$, and for $v \neq 0$

$$\frac{\lambda_1||X_{\mathrm{A}}v||^2}{||X_{\mathrm{A}}v||^2 + ||v||^2} \geqslant \frac{\lambda_1\gamma^2}{\gamma^2+1} =: \lambda_0' > 0.$$

Thus (11.1) holds with $\lambda_0 = \lambda_0'$.

In case (b), we have

$$W_{\mathrm{l}} \geqslant \lambda_{\min}(V_{\tilde{b}})\ ||v||^2 \quad \text{and} \quad W_{\mathrm{r}} \leqslant \lambda_0\ ||v||^2(\gamma^2+1).$$

Then for $\lambda_0 = \lambda_0'' := \lambda_{\min}(V_{\tilde{b}})/(\gamma^2+1)$, inequality (11.1) holds. In order to cover both cases (a) and (b), we can set

$$\lambda_0 := \min(\lambda_0', \lambda_0'') = \frac{1}{\gamma^2+1}\min(\lambda_1\gamma^2, \lambda_{\min}(V_{\tilde{b}})) > 0,$$

and then (11.1) is proved.

Thus Theorems 3 and 4 are applicable, if at least one of conditions (A) or (B) hold. We mention that in an univariable case $d=1$, condition (B) is equivalent to the following condition:

$$\inf_{\substack{X\in\Theta_X,\ X_A\neq 0\\ \omega\in\mathbb{T}}} \frac{1}{||X_A||^2}\left(X_A^\top\left(\sum_{k=-n_s+1}^{n_s+1} V_{\tilde{a},k}\omega^k\right)X_A\right) > 0.$$

Also, under condition (A), the moment assumption (viii) can be relaxed. namely, it is enough to demand that (5.11) holds for certain fixed real $\gamma \geqslant 2$, $\gamma > dn_s$. The reason is that under condition (A), the set $\{X_A : X \in \Theta_X\}$ is already compact, and the regularized problem (5.9) is not needed. The proof of Theorem 3 now goes in the same line as in Sections 8 and 9, and Appendix A but in terms of X rather then Z. The modified moment condition is used, e.g., to bound the corresponding $U_1(X)$ and $U_2(X)$. To bound $U_{11}(X_A)$ (compare with Section 3)we need $\gamma > dn_s$, where dn_s is the dimension of $\{X_A \in \mathbb{R}^{d\times n_s} : ||X_A||_F \leqslant L\}$.

12. Applications

The STLS problem formulation, given in Section 2, covers a wide number of applications. In particular, discrete-time linear dynamical models can be described by a structured system of linear equations and subsequently the identification problem rephrased as an STLS problem. We show three examples of linear discrete-time system identification problems that reduce to the STLS problem described in the paper.

12.1. FIR system impulse response estimation

Consider the finite impulse response (FIR) system

$$\bar{y}(t) = \bar{h}(0)\bar{u}(t) + \cdots + \bar{h}(q)\bar{u}(t-q+1).$$

Here $\bar{h}$ is the system *impulse response*, $\bar{u}$ is the *exact input*, and $\bar{y}$ is the *exact output*. Noisy measurements u and y are obtained from the true input/output signals $\bar{u}$ and $\bar{y}$, i.e.,

$$u = \bar{u} + \tilde{u} \quad \text{and} \quad y = \bar{y} + \tilde{y}$$

where $\tilde{u}$ and $\tilde{y}$ are zero-mean random signals with i.i.d. samples. The problem we are interested in is: given the measurements (u, y), estimate the system impulse response $\bar{h}$. The impulse response length q is assumed known.

Over a finite horizon $0, 1, \ldots, t_f - 1$, the response of the system can be written as a matrix–vector multiplication:

$$\begin{bmatrix} \bar{u}(0) & \bar{u}(-1) & \cdots & \bar{u}(-q+1) \\ \bar{u}(1) & \bar{u}(0) & \cdots & \bar{u}(-q+2) \\ \vdots & \vdots & \ddots & \vdots \\ \bar{u}(t_f-1) & \bar{u}(t_f-2) & \cdots & \bar{u}(0) \end{bmatrix} \begin{bmatrix} \bar{h}(0) \\ \bar{h}(1) \\ \vdots \\ \bar{h}(q-1) \end{bmatrix} = \begin{bmatrix} \bar{y}(0) \\ \bar{y}(1) \\ \vdots \\ \bar{y}(t_f-1) \end{bmatrix}. \tag{12.1}$$

(The negative time samples $\bar{u}(-1), \ldots, \bar{u}(-q+1)$ are the system initial conditions, which are also assumed measured with additive noise.) System (12.1) is satisfied for the exact input/output signals and the exact impulse response. If we replace the exact signals with the measured ones, (12.1) defines an STLS problem for the estimation of the unknown impulse response. The problem is with a Toeplitz structured A matrix and unstructured B matrix. The STLS estimate $\hat{h}$ provides a consistent estimator of the FIR system impulse response $\bar{h}$.

12.2. ARMA model identification

Consider the ARMA system

$$\sum_{\tau=0}^{q} \bar{y}(t+\tau)a(\tau) = \sum_{\tau=0}^{q} \bar{u}(t+\tau)b(\tau). \tag{12.2}$$

As before, $\bar{y}$ is the *exact output* and $\bar{u}$ is the *exact input*. We measure the input and the output with additive noises

$$y = \bar{y} + \tilde{y}, \quad u = \bar{u} + \tilde{u}$$

and assume that the noise samples $\tilde{y}(t)$ and $\tilde{u}(t)$ are zero mean i.i.d. The problem is to estimate the parameters

$$a := [a(q) \;\; a(q-1) \;\; \cdots \;\; a(0)], \quad \text{and} \quad b := [b(q) \;\; b(q-1) \;\; \cdots \;\; b(0)]$$

from a given set of input/output measurements $\{u(t), y(t)\}_{t=0}^{t_f-1}$. We assume that the order q of the model is known.

This identification problem is naturally expressed as an STLS problem; moreover the resulting problem satisfies our assumptions. From (12.2), we have

$$\sum_{\tau=0}^{q} \bar{u}(t+\tau)b(\tau) + \sum_{\tau=0}^{q-1} -\bar{y}(t+\tau)a(\tau) = \bar{y}(t+q)a(q).$$

The parameters can be normalized by setting $a(q) = 1$. Then for a time horizon $t = 0, 1, \ldots, t_f - 1$, the system equations are written as a structured linear system of equations:

$$\left[\begin{array}{cccc|cccc} \bar{u}(0) & \bar{u}(1) & \cdots & \bar{u}(q) & \bar{y}(0) & \bar{y}(1) & \cdots & \bar{y}(q-1) \\ \bar{u}(1) & \bar{u}(2) & \cdots & \bar{u}(q+1) & \bar{y}(1) & \bar{y}(2) & \cdots & \bar{y}(q) \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{u}(m) & \bar{u}(m+1) & \cdots & \bar{u}(m+q) & \bar{y}(m) & \bar{y}(m+1) & \cdots & \bar{y}(m+q-1) \end{array}\right] \left[\begin{array}{c} b(0) \\ \vdots \\ b(q) \\ \hline -a(0) \\ \vdots \\ -a(q-1) \end{array}\right] = \begin{bmatrix} \bar{y}(q) \\ \bar{y}(1+q) \\ \vdots \\ \bar{y}(m+q) \end{bmatrix}, \tag{12.3}$$

where $m = t_f - 1 - q$. The number of estimated parameters is $n = 2q + 1$. We assume that the time horizon is large enough to ensure $m \gg n$. System (12.3) is satisfied for the exact input and the exact output. The unique solution is given by the true values of the parameters.

If we replace the exact input/output data with the measured one, in general system (12.3) is no longer compatible. The STLS approach can be applied to simultaneously correct the measurements and *consistently* estimate the parameters. The resulting problem has structured $[A \; B]$ matrix, where the structure is given by two Hankel blocks. If the input is measured without additive noise, then the structure is given by a Hankel block and a noise-free block.

12.3. Hankel low-rank approximation

Let $H : \mathbb{R}^{n+m-1} \to \mathbb{R}^{m \times n}$, $m > n$, be a function that constructs an $m \times n$ Hankel matrix out of its parameters (the elements in the first column and the last row). The problem we consider is: given a full rank Hankel matrix $H(p)$, find a singular Hankel matrix $H(\hat{p})$, such that $||p - \hat{p}||_2^2$ is minimal. Thus $H(\hat{p})$ is the best (in the above-specified sense) low rank approximation of $H(p)$. This abstractly defined problem is related to the linear system realization and the model reduction problems in system theory (Kailath, 1981).

The Hankel low-rank approximation problem can be rephrased as an STLS problem. Indeed the problem is equivalent to

$$\min_{\hat{x}, \hat{p}} ||p - \hat{p}||_2^2 \quad \text{s.t.} \quad H(\hat{p}) \begin{bmatrix} \hat{x} \\ -1 \end{bmatrix} = 0, \tag{12.4}$$

because the constraint ensures that the nonzero vector $(\hat{x}, -1)$ is in the null space of $H(\hat{p})$. Clearly, (12.4) is an STLS problem with Hankel structured $[A \; B]$ matrix. Note that here the structure parameter estimate $\hat{p}$ (and not $\hat{x}$) is of interest.

The solution $\hat{p}$ of problem (12.4) guarantees that $H(\hat{p})$ is singular. If the rank of $H(\hat{p})$ has to be lowered by more than one (this is of interest for the model reduction problem), then a problem with multiple right-hand sides should be considered. With $\hat{X} \in \mathbb{R}^{n\times d}$, the solution of

$$\min_{\hat{X},\hat{p}} \|p-\hat{p}\|_2^2 \quad \text{s.t.} \quad H(\hat{p}) \begin{bmatrix} \hat{X} \\ -I \end{bmatrix} = 0,$$

ensures that $\operatorname{rank}(H(\hat{p})) \leqslant n-d$. (By assumption (vi), $\hat{X}$ is of full rank.)

In the realization and the model reduction problems, the Hankel matrix $H(p)$ is composed of measurements of the Markov parameters of the system. For single-input/single-output systems they are scalars, but in the multivariable case they become matrices with dimension (number of outputs)$\times$(number of inputs). This motivates an extension of the results of the paper for block Toeplitz/Hankel structured problems. Such a generalization is described in the next section.

13. A model with block-Hankel/Toeplitz structure

Consider model (1.2). We fix a natural number $q \geqslant 2$, the size of the blocks (assumed square), and suppose that $m = ql$, where $l \in \mathbb{N}$. Denote

$$A^\top =: [A_1 \cdots A_l] \quad \text{and} \quad B^\top =: [B_1 \cdots B_l],$$

where $A_i \in \mathbb{R}^{n\times q}$ and $B_i \in \mathbb{R}^{d\times q}$, $i = 1, \ldots, l$. We use the same notation for the blocks of $\tilde{A}$, A_0, $\tilde{B}$, B_0, A_s, and B_s.

Next, we introduce the new assumptions. We still assume that conditions (i)–(iii) hold.

(iv)$'$ The sequence $\{[\tilde{A}_{\mathrm{s}i}^\top \ \tilde{B}_{\mathrm{s}i}^\top]^\top,\ i = 1, 2, \ldots, l\}$ is stationary in a wide sense and s-dependent.

A centered sequence $\{D_i,\ i = 1, 2, \ldots\}$ of random matrices is called stationary in a wide sense if the sequence $\{\operatorname{vec}(D_i),\ i = 1, 2, \ldots\}$ is stationary in a wide sense, see Section 2.

We still assume that conditions (v) and (vi) hold. Now, we state an analogue of assumption (vii). Consider the matrix Γ, see (3.4). It consists of the blocks

$$F_{ij} := \mathbf{E} \operatorname{vec}(X_{\mathrm{ext}}^\top \tilde{C}_i) \operatorname{vec}^\top (X_{\mathrm{ext}}^\top \tilde{C}_j) \in \mathbb{R}^{dq\times dq}, \quad \text{for } i, j = 1, \ldots, l.$$

Here $\tilde{C}^\top =: [\tilde{C}_1 \cdots \tilde{C}_l]$, where $\tilde{C}_i \in \mathbb{R}^{(n+d)\times q}$ and according to assumption (iv)$'$, $\{\tilde{C}_i : i = 1, 2, \ldots, l\}$ form a wide sense stationary and s-dependent sequence. Therefore $F_{ij} = F_{i-j}$ is a function of the difference $i-j$, and $F_{ij} = 0$ for such i, j that $|i-j| \geqslant n_\mathrm{s} + d_\mathrm{s}$. Then Γ has the block-banded structure (4.2) and $F_{-k} = F_k^\top$, for $k = 0, 1, \ldots, s$.

The function f_X, defined in (4.4), is related to Γ. A new condition is stated in terms of $f_X(\omega)$, however.

(vii)$'$ There exists $\lambda_0 > 0$, such that for each $X \in \Theta_X$, $v \in \mathbb{R}^{dq\times 1}$, and $\omega \in \mathbb{T}$,

$$v^\top f_X(\omega) v \geqslant \lambda_0 \left\| \begin{bmatrix} X_{\mathrm{s}} v_1 \\ \vdots \\ X_{\mathrm{s}} v_q \end{bmatrix} \right\|^2,$$

where $v =: [v_1^\top \cdots v_q^\top]^\top$, with $v_i \in \mathbb{R}^{d\times 1}$, $i = 1, \ldots, q$.

Under conditions (vi) and (vii)$'$, Γ is non-singular, and $\sigma(\Gamma) \subset \bigcup_{\omega\in\mathbb{T}} \sigma(f_X(\omega)) \subset (0, \infty)$.

We state the consistency results similar to Theorems 3 and 4.

Theorem 7. *Assume that conditions* (i)–(iii), (iv)$'$, (v), (vi), (vii)$'$, *and* (viii)–(x) *hold. Then, the STLS estimator* $\hat{X}$ *converges in probability to the true value* X_0, as $l := m/q$ *goes to infinity, i.e.,*

$$\hat{X} \xrightarrow{\mathrm{P}} X_0, \quad as\ l \to \infty.$$

Moreover,

$$\|\hat{X}_m - X_0\|_{\mathrm{F}} = \mu_m \cdot \mathrm{O_p}(1), \quad for\ m = lq,\ l \in \mathbb{N},$$

where μ_m *is defined in* (7.1).

Theorem 8. *In the assumptions of Theorem* 7, *replace* (ix) *and* (x) *with* (xi) *and* (xii). *Then*

$$\hat{X} \to X_0, \quad as\ l = m/q \to \infty,\ a.s.$$

Theorems 7 and 8 are applicable for the block-Hankel/Toeplitz structured $[A\ \ B]$. e.g.,

$$[A\ \ B] = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \\ \vdots & \vdots \\ P_l & P_{l+1} \end{bmatrix}, \quad \text{where} P_i \in \mathbb{R}^{q\times q}$$

and there is no additional structure inside P_i.

14. Conclusions

Consistency results for the STLS estimator were presented. An affine structure was considered, with the additional assumptions that the errors in the measurements are stationary in a wide sense and s-dependent for certain $s \in \mathbb{N}$. The assumptions are mild; for example, they allow Toeplitz/Hankel structured, unstructured, and error-free blocks together in the augmented data matrix.

An optimization problem equivalent to the one defining the STLS estimator was derived by analytic minimization over the nuisance parameters, defining the structure. The resulting problem $\min_{X\in\Theta_X} Q(X)$ has as decision variables only the estimated parameters. The cost function Q was used in the analysis. It has the structure $Q(X) = r^\top \Gamma^{-1} r$, where r is an affine function of X and the elements of Γ are quadratic functions of X. Under the s-dependence assumption, the matrix Γ is block banded. In addition, the blocks of Γ^{-1} have exponential decay, away from the main diagonal. This property was used to bound the quantity $Q(X) - Q(X_0)$, where X_0 is the true value of the estimated parameter. Based on the bound, weak and strong consistency of the STLS estimator were proven. The significance of the assumptions were illustrated with examples.

Based on the analysis of the cost function Q an iterative algorithm for the computation of the STLS estimator was proposed. The performance of the proposed algorithm is comparable with this of the currently best known STLS solvers. It is an open problem to show that the presented algorithm is a contraction.

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Appendix A. Bounds for the summands in $Q(Z) - \mathbf{E}\,Q(Z)$

We start with U_2 and use representation (6.10). Let $r_0 := [r_{01}^\top \cdots r_{0m}^\top]^\top$, where r_{0i} are defined in (8.2). Then

$$U_2 = \sum_{i,j=0}^{m} a_{0i}^\top \Delta V \Phi_{ij} \tilde{r}_j - r_0^\top \Gamma^{-1} D \tilde{r} =: U_{21} - U_{22}.$$

A.1. Bound for U_{21}

Let $\Delta V = [\Delta v_{jp}]_{j=1,\dots,n}^{p=1,\dots,d}$, and $\Delta V^\top =: [\Delta v_1 \cdots \Delta v_n]$.

We consider an elementary summand of U_{21}. For $Z^\top = [\, X^\top \;\; -I_d \,] = [\, z_1 \cdots z_{n+d} \,]$, we have

$$U_{21} = \sum_{i,j=1}^{m} \left(\sum_{j_1=1}^{n} a_{0,ij_1} \Delta v_{j_1}^\top \right) \Phi_{ij} \left(\sum_{j_2=n_\mathrm{f}+1}^{n_\mathrm{f}+s+1} z_{j_2} \delta_{jj_2} \right)$$

and

$$U_{21} = \sum_{\substack{1 \leqslant p,q \leqslant d,\;\; 1 \leqslant j_1 \leqslant n \\ n_\mathrm{f}+1 \leqslant j_2 \leqslant n_\mathrm{f}+s+1}} z_{qj_2} \Delta v_{j_1 p} \sum_{i,j=1}^{m} a_{0,ij_1} [\Phi_{ij}]_{pq} \delta_{jj_2}. \tag{A.1}$$

We consider a random field which is generic for an elementary summand of (A.1),

$$U'_{21} = U'_{21}(Z_\mathrm{s}) := \sum_{j=1}^{m} \delta_{jj_2} \sum_{i=1}^{m} a_{0,ij_1} [\Phi_{ij}]_{pq} \quad \text{for all } Z_\mathrm{s} \in \Theta_{Zs} \tag{A.2}$$

(Due to (5.2), Γ depends on Z through its component Z_s, therefore Φ_{ij} depends on Z through Z_s as well.) We apply Lemma 1 from Appendix B to bound $|U'_{21}(Z_\mathrm{s})|$.

Remember that $\{\delta_{jj_2},\;\; j=1,2,\dots\}$ is an s-dependent sequence so that we can use Lemma 2 (b). For γ from condition (viii), we have

$$\begin{aligned}
\mathbf{E}|U'_{21}|^{2\gamma} &\leqslant \mathrm{const} \left(\sum_{j=1}^{m} \left(\sum_{i=1}^{m} a_{0,ij_1} [\Phi_{ij}]_{pq} \right)^2 \right)^\gamma \\
&\leqslant \mathrm{const} \left(\sum_{j=1}^{m} \left(\sum_{i=1}^{m} |a_{0,ij_1}| \cdot ||\Phi_{ij}||_\mathrm{F} \right)^2 \right)^\gamma .
\end{aligned}$$

By (6.1), $||\Phi_{ij}||_\mathrm{F} \leqslant \mathsf{c}_1 \exp(-\mathsf{c}_2 \cdot |i-j|)$. Then

$$\mathbf{E}|U'_{21}|^{2\gamma} \leqslant \mathrm{const} \left(\sum_{j=1}^{m} \sum_{i_1,i_2=1}^{m} |a_{0,i_1j_1}| \cdot |a_{0,i_2j_1}| \exp(-\mathsf{c}_2 \cdot |i_1 - j|) \exp(\mathsf{c}_2 \cdot |i_2 - j|) \right)^\gamma .$$

We bound the sum in the brackets.

$$\sum_{j=1}^{m}\sum_{i_1,i_2=1}^{m} |a_{0,i_1 j_1}|\cdot|a_{0,i_2 j_1}|\exp(-\mathtt{c}_2\cdot|i_1-j|)\exp(\mathtt{c}_2\cdot|i_2-j|)$$
$$\leqslant \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\exp(-\mathtt{c}_2\cdot|k_1|)\exp(-\mathtt{c}_2\cdot|k_2|)\sum_{\substack{1\leqslant j\leqslant m\\ 1-k_1\leqslant j\leqslant m-k_1\\ 1-k_2\leqslant j\leqslant m-k_2}}|a_{0,k_1+j,j_1}|\cdot|a_{0,k_2+j,j_1}|$$
$$\leqslant \text{const}\sum_{i=1}^{m} a_{0,ij_1}^2 \leqslant \text{const}\ \lambda_{\max}(A_0^\top A_0).$$

Then

$$\mathbf{E}|U'_{21}|^{2\gamma}\leqslant \text{const}\ \lambda_{\max}^{\gamma}(A_0^\top A_0). \tag{A.3}$$

Now, we bound the moments of the increment of U'_{21}. There exists an $\varepsilon_0>0$ such that (compare with (5.7) and (5.8))

$$\inf_{\omega\in\mathbb{T}}\left(\inf_{\substack{Z_{\mathrm{s}},Y_{\mathrm{s}}\in\Theta_{Z_{\mathrm{s}}}\\ ||Z_{\mathrm{s}}-Y_{\mathrm{s}}||_{\mathrm{F}}\leqslant\varepsilon_0\\ \lambda+\mu=1,\ \ \lambda,\mu\geqslant 0}}\lambda_{\min}(f_{\lambda Z_{\mathrm{s}}+\mu Y_{\mathrm{s}}}(\omega))\right)\geqslant\frac{\lambda_0}{2}>0. \tag{A.4}$$

Here we used that the function f_Z, defined in (5.8), depends on Z through the component Z_{s}. Let $Z_{\mathrm{s}}, Y_{\mathrm{s}}\in\Theta_{Z_{\mathrm{s}}}$ and $||Z_{\mathrm{s}}-Y_{\mathrm{s}}||_{\mathrm{F}}\leqslant\varepsilon_0$. For an increments of Φ_{ij}, we have

$$|[\Phi_{ij}]_{pq}(Z_{\mathrm{s}})-[\Phi_{ij}]_{pq}(Y_{\mathrm{s}})|\leqslant\sup_{\substack{\bar Z_{\mathrm{s}}=\lambda Z_{\mathrm{s}}+\mu Y_{\mathrm{s}}\\ \lambda+\mu=1,\ \ \lambda,\mu\geqslant 0}}\left\|\frac{\partial[\Phi_{ij}]_{pq}(\bar Z_{\mathrm{s}})}{\partial Z_{\mathrm{s}}}\right\|\cdot||Z_{\mathrm{s}}-Y_{\mathrm{s}}||_{\mathrm{F}}. \tag{A.5}$$

This supremum has an exponential decay as $|i-j|$ grows. Indeed, from representation (6.3) we have

$$\frac{\partial[\Phi_{ij}]_{pq}(\bar Z_{\mathrm{s}})}{\partial Z_{\mathrm{s}}}=\frac{1}{2\pi\mathbf{i}}\oint_{|\omega|=1+\varepsilon/2}\frac{\partial}{\partial Z_{\mathrm{s}}}[f_{\bar Z_{\mathrm{s}}}^{-1}(\omega)]_{pq}\frac{\mathrm{d}\omega}{\omega^{i-j+1}}. \tag{A.6}$$

And due to (A.4)

$$\left\|\frac{\partial}{\partial Z_{\mathrm{s}}}[f_{\bar Z_{\mathrm{s}}}^{-1}(\omega)]_{pq}\right\|\leqslant\text{const},\quad \text{for all } |\omega|=1+\varepsilon/2$$

and

$$\bar Z_{\mathrm{s}}\in\{\lambda Z_{\mathrm{s}}+\mu Y_{\mathrm{s}}:\lambda+\mu=1,\ \ \lambda,\mu\geqslant 0,\ \ Z_{\mathrm{s}},Y_{\mathrm{s}}\in\Theta_{Z_{\mathrm{s}}}\}.$$

Therefore similarly to bound (6.1), we obtain from (A.6), that

$$\left\|\frac{\partial[\Phi_{ij}]_{pq}(\bar Z_{\mathrm{s}})}{\partial Z_{\mathrm{s}}}\right\|\leqslant\tilde{\mathtt{c}}_1\exp(-\tilde{\mathtt{c}}_2\cdot|i-j|),\quad \text{where } \tilde{\mathtt{c}}_1,\tilde{\mathtt{c}}_2>0.$$

We showed the exponential decay of the supremum in (A.5). From (A.5), we have

$$|[\Phi_{ij}]_{pq}(Z_s) - [\Phi_{ij}]_{pq}(Y_s)| \leqslant \text{const } \exp(-\tilde{c}_2 \cdot |i-j|) \cdot ||Z_s - Y_s||_F. \tag{A.7}$$

Using (A.2) and (A.7), we obtain by Lemma 2(b), see Appendix B,

$$\begin{aligned}
&\mathbf{E}|U'_{21}(Z_s) - U'_{21}(Y_s)|^{2\gamma} \\
&\quad \leqslant \text{const}\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{m}|a_{0,ij_1}|\cdot|[\Phi_{ij}]_{pq}(Z_s)-[\Phi_{ij}]_{pq}(Y_s)|\right)^2\right)^{\gamma} \\
&\quad \leqslant \text{const}\left(\sum_{j=1}^{m}\left(\sum_{i=1}^{m}|a_{0,ij_1}|\ \exp(-\tilde{c}_2\cdot|i-j|)\right)^2\right)^{\gamma}\cdot||Z_s - Y_s||_F^{2\gamma}.
\end{aligned}$$

Similarly to (A.3), this implies for $Z_s, Y_s \in \Theta_{Z_s}$, $||Z_s - Y_s||_F \leqslant \varepsilon_0$ that

$$\mathbf{E}|U'_{21}(Z_s) - U'_{21}(Y_s)|^{2\gamma} \leqslant \text{const}(\lambda_{\max}(A_0^\top A_0))^{\gamma} \cdot ||Z_s - Y_s||_F^{2\gamma}. \tag{A.8}$$

But Θ_{Z_s} is compact, therefore (A.8) holds for arbitrary $Z_s, Y_s \in \Theta_{Z_s}$.

Now, we apply Lemma 1, see Appendix B, to the stochastic field $U'_{21}(Z_s)$ on Θ_{Z_s} defined in (5.4), (5.5), with $p := d(n_s + d_s - (d+1)/2)$, though formally $\Theta_{Z_s} \in \mathbb{R}^{(n_s+d_s)\times d}$. It is possible to extend Lemma 1 to $\Theta_{Z_s} \subset \Theta_s$, see (5.6), using spherical coordinates and to apply Lemma 1 directly for these coordinates. Here $d(n_s + d_s - (d+1)/2)$ equals the dimension of Θ_s as a manifold, see (Kukush and Van Huffel, 2004, Section 6.2).

We mention that in (A.5) and (A.3), the exponent $2\gamma > d(n_s + d_s - (d+1)/2)$, due to assumption (viii). Therefore from Lemma 1, we have

$$\sup_{Z_s \in \Theta_{Z_s}} |U'_{21}(Z_s)| = \lambda_{\max}^{1/2}(A_0^\top A_0) \cdot \rho'_m, \tag{A.9}$$

where

$$\mathbf{P}\{\rho'_m > a\} \leqslant \text{const } a^{-2\gamma} \quad \text{for all } a > 0. \tag{A.10}$$

Now, from (A.9), (A.10), and (A.1), and taking into account that in (A.1), $|z_{qj_2}| \leqslant 1$, we have

$$\sup_{Z_s \in \Theta_{Z_s}} |U'_{21}(Z_s)| = \lambda_{\max}^{1/2}(A_0^\top A_0) \cdot ||\Delta V||_F \cdot \rho''_m, \tag{A.11}$$

where ρ''_m satisfies (A.10).

A.2. Bounds for U_{22} and U_2

We have, see (6.8), that

$$D\tilde{r} = \begin{bmatrix} \sum_{j=1}^{m} \beta_{1j}\tilde{r}_j \\ \cdots \\ \sum_{j=1}^{m} \beta_{s+1,j}\tilde{r}_j \\ 0 \\ \cdots \\ 0 \\ \sum_{j=1}^{m} \beta_{m-s,j}\tilde{r}_j \\ \cdots \\ \sum_{j=1}^{m} \beta_{mj}\tilde{r}_j \end{bmatrix}. \tag{A.12}$$

For U_{22}, we can consider an elementary summand, corresponding to the upper part of the right-hand side of (A.12),

$$U'_{22}(Z) := \sum_{i=1}^{m} r_{0i}^{\top} M_{ij_1} \sum_{j=1}^{m} \beta_{j_1 j}\tilde{r}_j, \quad \text{for all } 1 \leqslant j_1 \leqslant s+1.$$

We bound these two sums separately.

(a) $\left\| \sum_{i=1}^{m} r_{0i}^{\top} M_{ij_1} \right\|^2 \leqslant \sum_{i=1}^{m} ||a_{0i}^{\top} \Delta V||^2 \cdot \sum_{i=1}^{m} ||M_{ij_1}||_{\mathrm{F}}^2$. But

$$\sum_{i=1}^{m} ||M_{ij_1}||_{\mathrm{F}}^2 \leqslant d\lambda_{\max}^2(\Gamma^{-1}) \leqslant \text{const},$$

therefore

$$\left\| \sum_{i=1}^{m} r_{0i}^{\top} M_{ij_1} \right\|^2 \leqslant \text{const}\, \lambda_{\max}^{1/2}(A_0^{\top} A_0) \cdot ||\Delta V||_{\mathrm{F}}. \tag{A.13}$$

(b) Next, consider the field

$$U''_{22}(Z_{\mathrm{s}}) := \sum_{i=1}^{m} \beta_{j_1 j}\tilde{r}_j.$$

We have for γ, from condition (viii),

$$\mathbf{E}||U''_{22}||^{2\gamma} \leqslant \text{const}\left(\sum_{j=1}^{m} ||\beta_{j_1 j}||_{\mathrm{F}}^2 \right)^{\gamma} \leqslant \text{const}$$

due to (6.9). Similarly for $Z_{\mathrm{s}}, Y_{\mathrm{s}} \in \Theta_{Z_{\mathrm{s}}}$,

$$\mathbf{E}||U''_{22}(Z_{\mathrm{s}}) - U''_{22}(Y_{\mathrm{s}})||^{2\gamma} \leqslant \text{const}||Z_{\mathrm{s}} - Y_{\mathrm{s}}||_{\mathrm{F}}^{2\gamma}.$$

Then, like in (A.9),

$$\sup_{Z_{\mathrm{s}} \in \Theta_{Z_{\mathrm{s}}}} ||U''_{22}(Z_{\mathrm{s}})|| = \rho'''_m, \tag{A.14}$$

where ρ_m''' satisfies (A.10). Now from (A.13) and (A.14), we have for $Z \in \Theta_Z$, (we write it already for U_{22}, rather than for U_{22}')

$$|U_{22}(Z)| \leqslant \lambda_{\max}^{1/2}(A_0^\top A_0) \cdot ||\Delta V||_{\mathrm{F}} \cdot \tilde{\rho}_m, \tag{A.15}$$

where $\tilde{\rho}_m$ satisfies (5.1). Summarizing (A.11) and (A.15), we have

$$|U_2(Z)| \leqslant \lambda_{\max}^{1/2}(A_0^\top A_0) \cdot ||\Delta V||_{\mathrm{F}} \cdot \rho_m, \tag{A.16}$$

where ρ_m satisfies (5.1).

Now, we pass to

$$U_1 = \underbrace{\tilde{r}^\top \Phi \tilde{r} - \mathbf{E}\tilde{r}^\top \Phi \tilde{r}}_{U_{11}} - \underbrace{\tilde{r}^\top \Gamma^{-1} D\tilde{r}}_{U_{12}} + \underbrace{\mathbf{E}\tilde{r}^\top \Gamma^{-1} D\tilde{r}}_{U_{13}} =: U_{11} - U_{12} + U_{13}.$$

A.3. Bound for U_{11}

As $\Phi_{ij} = H_{i-j}$, we have

$$U_{11}(Z_{\mathrm{s}}) = \sum_{k=-m+1}^{m-1} \sum_{\substack{1 \leqslant j \leqslant m \\ 1-k \leqslant j \leqslant m-k}} (\tilde{r}_{k+j}^\top H_k \tilde{r}_j - \mathbf{E}\,\tilde{r}_{k+j}^\top H_k \tilde{r}_j).$$

For a random variable ξ, denote

$$||\xi||_\gamma := (\mathbf{E}|\xi|^\gamma)^{1/\gamma},$$

where γ comes from condition (viii). Denote also

$$a \vee b := \max(a,b) \quad \text{and} \quad a \wedge b := \min(a,b).$$

Then

$$||U_{11}||_\gamma \leqslant \sum_{k=-m+1}^{m-1} \left\| \sum_{j=1\vee(1-k)}^{m\wedge(m-k)} \tilde{r}_{k+j}^\top H_k \tilde{r}_j - \mathbf{E}\,\tilde{r}_{k+j}^\top H_k \tilde{r}_j \right\|_\gamma .$$

In the inner sum the variables are s-dependent. Then by Lemma 2(a), we have

$$||U_{11}||_\gamma \leqslant \sum_{k=-m+1}^{m-1} \text{const } \max \left(\sum_{j=1\vee(1-k)}^{m\wedge(m-k)} \mathbf{E}(\tilde{r}_{k+j}^\top H_k \tilde{r}_j - \mathbf{E}\tilde{r}_{k+j}^\top H_k \tilde{r}_j)^2 \right.$$
$$\left. \times \left(\sum_{j=1\vee(1-k)}^{m\wedge(m-k)} \mathbf{E}|\tilde{r}_{k+j}^\top H_k \tilde{r}_j - \mathbf{E}\tilde{r}_{k+j}^\top H_k \tilde{r}_j|^\gamma \right)^{1/\gamma} \right).$$

From (6.1), we have for certain $\mathtt{c}_3 > 0$,

$$||U_{11}||_\gamma \leqslant \sum_{k=-\infty}^{\infty} \text{const}\sqrt{m}\, \exp(-\mathtt{c}_3 \cdot |k|) \leqslant \text{const}\sqrt{m}$$

and

$$\mathbf{E}|U_{11}|^{\gamma} \leqslant \text{const}\, m^{\gamma/2}.$$

Using (A.7), we have similarly, that for all $Z_{\rm s}, Y_{\rm s} \in \Theta_{Z_{\rm s}}$

$$\mathbf{E}|U_{11}(Z_{\rm s}) - U_{11}(Y_{\rm s})|^{\gamma} \leqslant \text{const}\, m^{\gamma/2} \cdot ||Z_{\rm s} - Y_{\rm s}||_{\rm F}^{\gamma}. \tag{A.17}$$

Due to our assumptions $\gamma > d(n_{\rm s} + d_{\rm s} - (d+1)/2)$, and $d(n_{\rm s} + d_{\rm s} - (d+1)/2)$ is the dimension of $\Theta_{\rm s}$ as a manifold, $\Theta_{\rm s} \supset \Theta_{Z_{\rm s}}$.

From (A.17), by Lemma 1, we have that,

$$\sup_{Z_{\rm s} \in \Theta_{Z_{\rm s}}} |U_{11}(Z_{\rm s}) - U_{11}(Z_{0\rm s})| = \sqrt{m}\,\xi'_m,$$

where $Z_{0\rm s}$ is $Z_{\rm s}$-component of $Z = Z_0$, and

$$\mathbf{P}\{\xi'_m > a\} \leqslant \text{const}\ a^{-\gamma} \quad \text{for all } a > 0.$$

A.4. Bound for U_{12}

Due to the structure of D, it is enough to consider an elementary summand of U_{12},

$$U'_{12} := \sum_{i=1}^{m} \tilde{r}_i M_{ij_1} \sum_{j=1}^{m} \beta_{j_1 j} \tilde{r}_j \quad \text{for all } 1 \leqslant j_1 \leqslant s+1. \tag{A.18}$$

(The corresponding summands for $m - s \leqslant j_1 \leqslant m$ can be treated similarly.) The second factor in (A.18) is denoted above by $U''_{22}(Z_{\rm s})$, and it permits the bound (A.14). Let

$$U''_{12}(Z_{\rm s}) := \sum_{i=1}^{m} \tilde{r}_i M_{ij_1}, \quad \text{for all } Z_{\rm s} \in \Theta_{Z_{\rm s}} \subset \Theta_{\rm s}.$$

For γ from condition (viii), we have due to (6.1) and Lemma 2(b) from Appendix B,

$$\mathbf{E}||U''_{12}(Z_{\rm s})||^{2\gamma} \leqslant \text{const} \sum_{i=1}^{m} \exp(-\tilde{c}_2|i - j_1|) \leqslant \text{const}$$

and for all $Z_{\rm s}, Y_{\rm s} \in \Theta_{Z_{\rm s}}$

$$\mathbf{E}||U''_{12}(Z_{\rm s}) - U''_{12}(Y_{\rm s})||^{2\gamma} \leqslant \text{const}||Z_{\rm s} - Y_{\rm s}||_{\rm F}^{2\gamma}.$$

Therefore

$$\sup_{Z_{\rm s} \in \Theta_{Z_{\rm s}}} ||U''_{12}(Z_{\rm s})|| = \tilde{\tilde{\rho}}_m,$$

where $\tilde{\tilde{\rho}}_m$ satisfies (A.10). Then

$$\sup_{Z_{\rm s} \in \Theta_{Z_{\rm s}}} |U'_{12}(Z_{\rm s})| \leqslant \sup_{Z_{\rm s} \in \Theta_{Z_{\rm s}}} ||U''_{12}(Z_{\rm s})|| \sup_{Z_{\rm s} \in \Theta_{Z_{\rm s}}} ||U''_{22}(Z_{\rm s})|| = \tilde{\tilde{\rho}}_m \rho'''_m =: \xi''_m$$

and

$$\mathbf{P}\{\xi_m'' > a\} \leqslant \mathbf{P}\{\tilde{\tilde{\rho}}_m > \sqrt{a}\} + \mathbf{P}\{\rho_m''' > \sqrt{a}\} \leqslant \text{const}(\sqrt{a})^{-2\gamma}$$
$$= \text{const } a^{-\gamma} \quad \text{for } a > 0.$$

A.5. Bounds for U_{13} and U_1

It is enough to consider an elementary summand for $1 \leqslant j_1 \leqslant s+1$

$$U_{13}'(Z_{\mathrm{s}}) := \mathbf{E} \sum_{i=1}^{m} \tilde{r}_i^{\top} M_{ij_1} \sum_{j=1}^{m} \beta_{j_1 j} \tilde{r}_j.$$

But $\{\tilde{r}_j\}$ are s-dependent. Therefore,

$$U_{13}' = \sum_{\substack{1 \leqslant i,\ j \leqslant m \\ |i-j| \leqslant s+1}} \mathbf{E}\, \tilde{r}_i^{\top} M_{ij_1} \beta_{j_1 j} \tilde{r}_j.$$

Recall that $||\beta_{j_1 j}||_{\mathrm{F}} \leqslant \text{const } \exp(-\mathrm{c}_2 \cdot j)$, $1 \leqslant j_1 \leqslant s+1$, we have

$$|U_{13}'| \leqslant \text{const} \sum_{j=-\infty}^{\infty} \exp(-\mathrm{c}_2 |j|) = \text{const}.$$

Thus $|U_{13}| \leqslant \text{const}$. Summarizing, we have,

$$\sup_{Z_{\mathrm{s}} \in \Theta_{Z_{\mathrm{s}}}} |U_1(Z_{\mathrm{s}}) - U_1(Z_{0\mathrm{s}})| = \sqrt{m}\,\xi_m, \tag{A.19}$$

where

$$\mathbf{P}\{\xi_m > a\} \leqslant \text{const} \cdot a^{-\gamma}, \quad \text{for } a > 0.$$

Appendix B. Results on stochastic fields and moment inequalities

We recall the following result on stochastic fields (Ibragimov and Hasminskii, 1981).

Lemma 1. *Let $T(\beta)$ be a separable, measurable stochastic field defined on the compact set $K \subset \mathbb{R}^p$. Suppose that for any β, $\tilde{\beta}$ (for which $\beta \in K$, $\beta + \tilde{\beta} \in K$)*

$$\mathbf{E}|T(\beta + \tilde{\beta}) - T(\beta)|^{\gamma} \leqslant L||\tilde{\beta}||^{\gamma'}$$

for some $\gamma \geqslant \gamma' > p$ positive constant L. Then for any $a > 0$,

$$\mathbf{P}\{L^{-1/\gamma} \sup_{\beta_1,\ \beta_2 \in K} |T(\beta_1) - T(\beta_2)| > a\} \leqslant k_0 a^{-\gamma},$$

where k_0 depends on γ, γ', p, and K but does not depend on L and a.

We give also the following version of the Rosenthal inequality (Rosenthal, 1970).

Lemma 2. *Let* $\{v_k,\ k \geqslant 1\}$ *be a sequence of q-dependent random vectors in* $\mathbb{R}^p$, $\mathbf{E}v_k = 0$, *for all* $k \geqslant 1$. *Then*

(a) *for each* γ, $\gamma \geqslant 2$, *and for all* $m \geqslant 1$, *the following inequality holds*:

$$\mathbf{E}\left\|\sum_{k=1}^{m} v_k\right\|^{\gamma} \leqslant \mathtt{c}_1 \max\left\{\left(\sum_{k=1}^{m} \mathbf{E}||v_k||^2\right)^{\gamma/2}, \sum_{k=1}^{m} \mathbf{E}||v_k||^{\gamma}\right\},$$

(b) *for each real* γ, $\gamma \geqslant 2$, *for every* $m \geqslant 1$, *and* $\{a_1, \ldots, a_m\} \subset \mathbb{R}$, *the following inequality holds*:

$$\mathbf{E}\left\|\sum_{k=1}^{m} a_k v_k\right\|^{\gamma} \leqslant \mathtt{c}_2 \left(\sum_{k=1}^{m} a_k^2\right)^{\gamma/2} \sup_{1 \leqslant k \leqslant m} \mathbf{E}||v_k||^{\gamma},$$

where $\mathtt{c}_1$ *and* $\mathtt{c}_2$ *depend on* γ, *p, and q but do not depend on m and the choice of* $a_1, \ldots, a_m$.

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