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SKELETON APPROXIMATIONS OF OPTIMAL STOPPING STRATEGIES FOR AMERICAN TYPE OPTIONS WITH CONTINUOUS TIME[1,2]

American type options are studied for continuous pricing processes. The skeleton type approximations are considered. The explicit upper bounds are given for the step of discretisation for ε-optimal stopping strategies.

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1. Introduction

Traditional methods of option pricing are based on models of pricing processes which are various modifications of the classical model of geometrical Brownian motion. Stochastic differential equations can be written down for such pricing processes. Then partial differential equations and the corresponding variational problems can be derived for functions which represent optimal strategies, see for instance Øksendal (1992), Duffie (1996) and Karatzas and Shreve (1998). Finally various numerical algorithms can be applied to find optimal strategies for continuous time models and their discrete time approximations. The extended survey of latest results can be found in the book edited by Rogers and Talay (1998), in particular in the paper by Broadie and Detemple (1998).

We do prefer to use an alternative approach for evaluation of optimal stopping Buyer strategies for American type options. The structure of optimal stopping strategies is investigated by applying of the direct probabilistic analysis under general assumptions for underlying pricing processes.

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In the papers by Kukush and Silvestrov (2000a, 2000b) the structure of optimal stopping strategies were investigated for a general model of discrete time pricing processes and pay-off functions. The model of pricing process is a two component inhomogeneous in time Markov process with a phase space $[0, \infty) \times Y$. The first component is the corresponding pricing process and the second component (with a general measurable phase space Y) represents some stochastic index process controlling the pricing process. Pay-off functions under consideration are in sequel: (a) an inhomogeneous in time analogue of a standard one $g_n(x) = a_n[x - K_n]_+$; (b) piecewise linear convex functions, and finally (c) general convex functions.

At present paper we study skeleton type approximations for continuous time pricing processes. The explicit upper bounds are given for the step of discretisation for ε-optimal stopping strategies. These upper bounds enable us to use the results given in Kukush and Silvestrov (2000a, 2000b) for constructive description of ε-optimal stopping strategies for American type options with continuous time. The special attention is paid to the case of general model of pricing processes which are geometrical diffusion processes controlled by stochastic index processes.

We think that the main advantage of direct probabilistic approach in structural studies of optimal stopping strategies is that this approach is much more flexible and less sensitive to the modifications of models of underlying pricing processes, pay-off functions and other characteristics of the models.

The knowledge of the explicit structure of optimal stopping strategies is the base for the creation of effective optimising Monte Carlo pricing algorithms for numerical evaluation of the corresponding optimal strategies. Such algorithms and programs have been elaborated by Silvestrov, Galochkin and Sibirtsev (1999). We would like also to refer to the papers by

We would like to refer to the book by Shiryaev (1978) and the paper by Shiryaev, Kabanov, Kramkov, and Mel'nikov (1994), which stimulated the present research. We also refer to the paper by Kukush and Silvestrov (1999), where part of the current results was presented without the proofs.

2. Skeleton approximations for American type options in continuous time

Consider a two component inhomogeneous in time Markov process $Z_t = (S_t, I_t)$, $t \geq 0$, with a phase space $Z = [0, \infty) \times Y$. Here $(Y, \mathcal{B}_Y)$ is a general measurable space and as usual we consider Z as a measurable space with the σ–field $\mathcal{B}_Z = \sigma(\mathcal{B}_+ \times \mathcal{B}_Y)$ where $\mathcal{B}_+$ is a Borel σ–field on $R^+ = [0, \infty)$.

We assume that Z_t, $t \geq 0$ is a measurable process ($Z_t(\omega)$, $t \geq 0$ are $\mathcal{B}_Z$–measurable functions with respect to (t, ω)). Without loss of generality we assume that $Z_0 = (S_0, I_0)$ is a non-random value in Z.

We interpret the first component S_t as a pricing process and the second component I_t as a stochastic index process controlling the pricing process.

A basic example of the model described above is the pricing process given in the following form:

$$S_t = S_0 \cdot \exp\{\int\limits_0^t (a(u, I_u) - \frac{1}{2}\,\sigma(u, I_u)^2)du + \int\limits_0^t \sigma(u, I_u)dw(u)\},\ t \geq 0,$$

where (a) $a(t, y)$ and $\sigma(t, y) \geq 0$ are measurable real-valued functions defined on Z, (b) $I_t,\ t \geq 0$ is a measurable inhomogeneous in time Markov process such that functions $E|a(t, I_t)|$ and $E\sigma(t, I_t)^2$ are integrable at finite intervals and $w(u), u \geq 0$ is the Wiener process independent of process $I_t,\ t \geq 0$, (d) $Z_0 = (S_0, I_0)$ is a non-random value in Z.

In this case vector process $Z_t = (S_t, I_t), t \geq 0$ is an inhomogeneous Markov process with the first component $S_t, t \geq 0$ is a continuous geometrical diffusion process controlled by the index process $I_t, t \geq 0$.

Let $\mathcal{F}_t,\ t \geq 0$ be a flow of σ–fields, associated with process $Z_t,\ t \geq 0$. We shall consider Markov moments τ with respect to $\mathcal{F}_t$. It means that τ is a random value distributed in $[0, \infty]$ and with the property $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t, t \geq 0$.

Introduce further a pay-off function $g(x, t), x \in R^+, t \geq 0$. We assume that $g(x, t)$ is a nonnegative measurable function. Let also $R_t, t \geq 0$ be a nondecreasing function with $R_0 = 0$. Typically $R_t = \int_0^t r(s)ds$, where $r(s) \geq 0$ is a Borel function representing riskless interest rate at moment s.

The typical example of pay-off function is:

$$g(x, t) = a_t\,[x - K_t]_+ = \begin{cases} a_t\,(x - K_t), & \text{if } \ x > K_t, \\ 0, & \text{if } \ 0 \leq x \leq K_t, \end{cases}$$

where $a_t, t \geq 0$ and $K_t, t \geq 0$ are two nonnegative measurable functions. The case, where $a_t = a$ and $K_t = K$ do not depend on t, corresponds to the standard American call option.

We fix parameter $T > 0$ which we call an expiration date. It is convenient to operate with the transformed pricing process $S_g(t) = e^{-R_t}g(S_t, t), t \geq 0$. Let us formulate conditions which we impose on pricing processes and pay-off functions:

A: $S_g(t),\ t \geq 0$ is a.s. continuous from the right process.

B: $E \sup\limits_{0 \leq t \leq T} S_g(t) < \infty$.

Let denote $\mathcal{M}_{max,T}$ the class of all Markov moments $\tau \leq T$. Let now choose a partition $\Pi = \{0 = t_0 < t_1 < \dots t_N = T\}$ of interval $[0, T]$. We also consider the class $\hat{\mathcal{M}}_{\Pi,T}$ of all Markov moments from $\mathcal{M}_{max,T}$ which only take the values $t_0, t_1, \dots t_N$, and the class $\mathcal{M}_{\Pi,T}$ of all Markov moments from $\hat{\mathcal{M}}_{\Pi,T}$ such that event $\{\omega : \tau(\omega) = t_k\} \in \sigma[Z_0, \dots Z_{t_k}]$ for $k = 0, \dots N$. By definition

$$\mathcal{M}_{\Pi,T} \subseteq \hat{\mathcal{M}}_{\Pi,T} \subseteq \mathcal{M}_{max,T}. \tag{1}$$

The goal functional under consideration is:

$$\Phi_g(\tau) = Ee^{-R_\tau} g(S_\tau, \tau). \tag{2}$$

Denote for a class of Markov moments $\mathcal{M}_T \subseteq \mathcal{M}_{max,T}$

$$\Phi_g(\mathcal{M}_T) = \sup_{\tau \in \mathcal{M}_T} Ee^{-R_\tau} g(S_\tau,\, \tau). \tag{3}$$

Conditions **A**, **B** and relation (1) imply that

$$\Phi_g(\mathcal{M}_{\Pi,T}) \leq \Phi_g(\hat{\mathcal{M}}_{\Pi,T}) \leq \Phi_g(\mathcal{M}_{max,T}) < \infty. \tag{4}$$

Random variables $Z_{t_0}, Z_{t_1}, \ldots Z_{t_N}$ are connected in an inhomogeneous Markov chain with discrete time. The optimisation problem (2)-(3) for the class $\mathcal{M}_{\Pi,T}$ is a problem of optimal pricing for American type options with discrete time.

In Kukush and Silvestrov (2000a, 2000b) the structure of optimal and ε-optimal stopping moments in the class $\mathcal{M}_{\Pi,T}$ is described for various classes of convex in x pay-off functions $g(x, t_k), k = t_0, \ldots t_N$. Also, optimising Monte Carlo algorithms and programs for numerical evaluation of optimal stopping strategies, functionals $\Phi_g(\mathcal{M}_{\Pi,T})$ and other functionals for standard American options with discrete time are described in Silvestrov, Galochkin and Sibirtsev (1999).

Our goal is to show in which way the functional $\Phi_g(\mathcal{M}_{max,T})$ can be approximated by functionals $\Phi_g(\mathcal{M}_{\Pi,T})$ and to give explicit upper bounds for the accuracy of this approximation. This makes it possible to find stopping moments $\tau_\varepsilon \in \mathcal{M}_{\Pi,T}$ that are 2ε-optimal stopping moments in the class $\mathcal{M}_{max,T}$.

The next important statement is a base of skeleton approximation.

Lemma 1. *For every partition* $\Pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ *of interval* $[0, T]$ *and for the classes* $\mathcal{M}_{\Pi,T}$ *and* $\hat{\mathcal{M}}_{\Pi,T}$ *of Markov moments*

$$\Phi_g(\mathcal{M}_{\Pi,T}) = \Phi_g(\hat{\mathcal{M}}_{\Pi,T}). \tag{5}$$

Proof. Consider the optimization problem (2)-(3) for the class $\hat{\mathcal{M}}_{\Pi,T}$ as a problem of optimal pricing for American type options with discrete time. For this purpose add to the random variables Z_{t_n} additional components $\bar{Z}_{t_n} = \{Z_t, t_{n-1} < t \leq t_n\}$ with the phase space $Z^{(t_{n-1},t_n]}$ endowed by cylindrical σ- field. Consider the extended Markov chain $I_n = (Z_{t_n}, \bar{Z}_{t_n})$. As is known (Shiryaev (1978)) the optimal stopping moment τ exists in any discrete time model, and it has the form of the first hitting time $\tau = \min\{0 \leq n \leq N : I_n \in D_n\}$, where optimal stopping domains D_n are determined by the transition probabilities of Markov chain I_n. However, in this case the transition probabilities depend only on values of the first component Z_{t_n}. This case was considered in the papers by Kukush and

Silvestrov (2000a, 2000b). To imbed the model described above in the model introduced in these papers one should to consider the two component Markov chain (S_n, I_n) with the components $S_n = S_{t_n}, I_n = (Z_{t_n}, \bar{Z}_{t_n})$. The first component S_n is in this case completely determined by the component $Z_{t_n} = (S_{t_n}, I_{t_n})$ while, as was pointed out above, transition probabilities of Markov chain I_n do depend only of the values of the first component Z_{t_n}. As was shown in Kukush and Silvestrov (2000a, 2000b) in this case the optimal stopping moment has the the form of the first hitting times for the process (S_{t_n}, Z_{t_n}) and do not depend on the component $\bar{Z}_{t_n}$. Since S_{t_n} is determined by Z_{t_n} this moment can by represented in the form $\tau = \min\{0 \leq n \leq N : Z_{t_n} \in D'_n\}$, i.e. as the first hitting time for the Markov Chain Z_{t_n}.

Therefore for the optimal stopping moment $\tau \in \mathcal{M}_{\Pi,T}$. Hence $\Phi_g(\mathcal{M}_{\Pi,T}) \geq \Phi_g(\hat{\mathcal{M}}_{\Pi,T})$, and by (4) we obtain equality (5). $\bigoplus$

For any Markov moment $\tau \in \mathcal{M}_{max,T}$ and a partition $\Pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ one can define the discretisation of this moment

$$\tau[\Pi] = \begin{cases} 0, & \text{if} \quad \tau = 0, \\ t_k, & \text{if} \quad t_{k-1} < \tau \leq t_k, \ k = 1, \ldots N. \end{cases}$$

Now, let τ_ε be ε-optimal stopping moment in the class $\mathcal{M}_{max,T}$, i.e. $ES_g(\tau_\varepsilon) \geq \Phi_g(\mathcal{M}_{max,T}) - \varepsilon$. Since $\tau_\varepsilon[\Pi] \in \hat{\mathcal{M}}_{\Pi,T}$ the relation (5) implies

$$ES_g(\tau_\varepsilon[\Pi]) \leq \Phi_g(\hat{\mathcal{M}}_{\Pi,T}) = \Phi_g(\mathcal{M}_{\Pi,T}) \leq \Phi_g(\mathcal{M}_{max,T}). \tag{6}$$

Denote $d(\Pi) = \max\{t_k - t_{k-1}, k = 1, \ldots N\}$. Let also $\Pi_N = \{0 = t_{0N} < t_{1N} < \ldots t_{NN} = T\}$ be a sequence of partitions such that $d(\Pi_N) \to 0$ as $N \to \infty$.

By definition $\tau_\varepsilon \leq \tau_\varepsilon[\Pi_N] \leq \tau_\varepsilon + d(\Pi_N)$. That is why condition **A** implies that random variables $S_g(\tau_\varepsilon[\Pi_N]) \to S_g(\tau_\varepsilon)$ as $N \to \infty$ almost surely. This relation, condition **B** and Lebesgue theorem easily implies that $ES_g(\tau_\varepsilon[\Pi_N]) \to ES_g(\tau_\varepsilon) \geq \Phi_g(\mathcal{M}_{max,T}) - \varepsilon$ as $N \to \infty$. Since ε can be chosen arbitrary small the last relation and (6) implies in an obvious way that under conditions **A** and **B**

$$\lim_{N\to\infty} \Phi_g(\mathcal{M}_{\Pi_N,T}) = \Phi_g(\mathcal{M}_{max,T}). \tag{7}$$

Relation (7) gives the base for the use of skeleton discrete time approximation for continuous time model. This relation guarantees that for any fixed $\varepsilon > 0$ there exists $N = N_\varepsilon$ such that $\Phi_g(\mathcal{M}_{max,T}) - \Phi_g(\mathcal{M}_{\Pi_{N_\varepsilon},T}) \leq \varepsilon$. Let τ'_ε be an ε-optimal stopping moment in the class $\mathcal{M}_{\Pi_{N_\varepsilon},T}$, i.e. $ES_g(\tau'_\varepsilon) \geq \Phi_g(\mathcal{M}_{\Pi_{N_\varepsilon},T}) - \varepsilon$. Obviously τ'_ε is a 2ε-optimal stopping moment in the class $\mathcal{M}_{max,T}$.

However, relation (7) does not give quantitative estimates which connect the maximal step of the partition $d(\Pi_N)$ with ε. Such estimates can be obtained with the use of inequality (6).

For a separable process $S(t), t \geq 0$ the modulus of continuity on the interval $[0, T]$ is defined in the following way:

$$\Delta_{h,T}(S(\cdot)) = \sup_{t',t'',\in[0,T],|t'-t''|\leq h} |S(t') - S(t'')|, h > 0.$$

Condition **B** implies that $E\Delta_{h,T}(S(\cdot)) < \infty$ for all $h > 0$. Note also that $E\Delta_{h,T}(S(\cdot))$ monotonically does not decrease in $h > 0$.

Let us assume the following condition:

C: $E\Delta_{h,T}(S_g(\cdot)) \to 0$ as $h \to 0$.

Under minimal assumption of separability of the process $S_g(t), t \geq 0$ condition **C** implies that this process is an a.s. continuous process. Therefore condition **A** holds.

Let τ_ε be ε-optimal stopping moment in the class $\mathcal{M}_{max,T}$. Then inequality (6) and the relation $\tau_\varepsilon \leq \tau_\varepsilon[\Pi_N] \leq \tau_\varepsilon + d(\Pi_N)$ imply that

$$\Phi_g(\mathcal{M}_{max,T}) - \Phi_g(\mathcal{M}_{\Pi_N,T}) \leq \varepsilon + ES_g(\tau_\varepsilon) - ES_g(\tau_\varepsilon[\Pi_N]) \leq \tag{8}$$

$$\leq \varepsilon + E\left|S_g(\tau_\varepsilon) - S_g\left(\tau_\varepsilon[\Pi_N]\right)\right| \leq \varepsilon + E\Delta_{d(\Pi_N),T}\big(S_g(\cdot)\big).$$

Since ε can be chosen arbitrary small relation (8) implies finally that

$$\Phi_g(\mathcal{M}_{max,T}) - \Phi_g(\mathcal{M}_{\Pi_N,T}) \leq E\,\Delta_{d(\Pi_N),T}\ \left(S_g\left(\cdot\right)\right). \tag{9}$$

Condition **C** implies that there exists $h = h_\varepsilon$ such that $E\Delta_{h_\varepsilon,T}(S_g(\cdot)) \leq \varepsilon$. Since $d(\Pi_N) \to 0$ as $N \to 0$ there exists $N = N_\varepsilon$ such that $d(\Pi_{N_\varepsilon}) \leq h_\varepsilon$. Let τ'_ε be an ε-optimal stopping moment in the class $\mathcal{M}_{\Pi_{N_\varepsilon},T}$. Then (9) implies that τ'_ε is a 2ε-optimal stopping moment in the class $\mathcal{M}_{max,T}$.

So, the problem is reduced to solving with respect to h the following inequality:

$$E\,\Delta_{h,T}\left(S_g(\cdot)\right) \leq \varepsilon. \tag{10}$$

In the next section we give explicit upper bounds for the expectation of the modulus of continuity $E\Delta_{h,T}(S_g(\cdot))$ in terms of moments of increments of the transformed pricing processes $S_g(t), t \geq 0$ and link explicitly the parameters h and ε.

3. Upper bounds for expectation of the modulus of continuity

Let $S(t), t \geq 0$ be a separable real-valued process. We assume that the following condition holds:

D: $E|S(t') - S(t'')|^m \leq H|t' - t''|^r, 0 \leq t', t'' \leq T$ for some $H > 0$ and $m, r > 1$.

We use estimates for tail probabilities for the modulus of continuity given in Gikhman and Skorokhod (1974). However, we estimate the expectation for the modulus of continuity and give the upper bounds with explicit constants due to detailed technical account at all steps of calculations.

Lemma 2. (Gikhman and Skorokhod (1974)). *Let $S(t), t \in [0,T]$ be a separable process, such that there exist nonnegative, nondecreasing function $g(h)$ and function $q(C,h)$, $C > 0$, $h > 0$, with*

$$P\{|S(t+h) - S(t)| > Cg(h)\} \leq q(C,h), \tag{11}$$

and

$$G = \sum_{n=0}^{\infty} g(T/2^n) < \infty, \quad Q(C) = \sum_{n=1}^{\infty} 2^n q(C, T/2^n) < \infty. \tag{12}$$

Then for each $\delta > 0$

$$P\{\sup_{0 \leq t', t'' \leq T} |S(t') - S(t'')| > \delta\} \leq Q(\delta/2G), \tag{13}$$

and for each $\varepsilon > 0$, $C > 0$

$$P\{\Delta_{\varepsilon,T}(S(\cdot)) > CG([\log_2 T/2\varepsilon])\} \leq Q([\log_2 T/2\varepsilon], C), \tag{14}$$

where

$$G(p) = \sum_{n=p}^{\infty} g(T/2^n), \quad Q(p,C) = \sum_{n=p}^{\infty} 2^n q(C, T/2^n). \tag{15}$$

Lemma 3. *Let condition* **D** *holds. Then $S(t)$, $t \in [0,T]$ is a.s. continuous process and for every $1 < r' < r$ and for every $0 < h \leq T$:*

$$E\,\Delta_{h,T}(S(\cdot)) \leq B_1 h^{\frac{r-1}{m}}, \tag{16}$$

where

$$B_1 = \frac{m}{m-1}\, 2^{\frac{r-1}{m}} (1 - 2^{-\frac{r'-1}{m}})^{-1} (1 - 2^{-(r-r')})^{-\frac{1}{m}} H^{\frac{1}{m}} T^{\frac{1}{m}}. \tag{17}$$

Proof. Apply Lemma 2 to the process $S(t)$. Fix a positive number $r' < r$ and set $g(h) = h^{\frac{r'-1}{m}}$. Find $G(p)$, $q(C,h)$ and $Q(p,C)$ defined in (15), (11) and (12).

$$G(p) = T^{\frac{r'-m}{m}}\, 2^{-\frac{p(r'-1)}{m}}\, (1 - 2^{-\frac{r'-1}{m}})^{-1}\,,$$

therefore

$$G\,([\log_2 \frac{T}{2\varepsilon}]) \le 2^{\frac{r'-1}{m}}\, \varepsilon^{\frac{r'-1}{m}}\, (1 - 2^{-\frac{r'-1}{m}})^{-1}.$$

We have by **D**:

$$P\{|S(t+h) - S(t)| > C\,g(h)\} \le$$
$$\frac{E\,|S(t+h) - S(t)|^m}{C^m\, g^m(h)} \le \frac{H}{C^m}\, h^{1+r-r'} \;:= q(C,h),$$

and

$$Q(p,C) = \frac{H\,T^{1+r-r'}}{C^m}\, 2^{-p(r-r')}\, (1 - 2^{-(r-r')})^{-1}.$$

Then

$$Q\,([\log_2 \frac{T}{2\varepsilon}],\, C) \le C^{-m}\, T\, H\, \varepsilon^{r-r'}\, (1 - 2^{-(r-r')})^{-1} \cdot 2^{r-r'}.$$

By (14) we obtain

$$P\{\Delta_{\varepsilon,T}\big(S(\cdot)\big) > C\, 2^{\frac{r'-1}{m}}\, \varepsilon^{\frac{r'-1}{m}}\, (1 - 2^{-\frac{r'-1}{m}})^{-1}\} \le$$
$$\le C^{-m}\, T\, H\, \varepsilon^{r-r'}\, (1 - 2^{-(r-r')})^{-1} \cdot 2^{r-r'}.$$

Denote

$$\delta = C\, 2^{\frac{r'-1}{m}}\, \varepsilon^{\frac{r'-1}{m}}\, (1 - 2^{-\frac{r'-1}{m}})^{-1}.$$

Then

$$P\{\Delta_{\varepsilon,T}(S(\cdot)) > \delta\} \le \frac{TkH\varepsilon^{r-1}}{\delta^m},$$

where

$$k = 2^{r-1}\, (1 - 2^{-\frac{r'-1}{m}})^{-m}\, (1 - 2^{-(r-r')})^{-1}.$$

Next,

$$E\,\Delta_{h,T}\,\big(S(\cdot)\big) = \int_0^\infty P\{\Delta_{h,T}\,\big(S(\cdot)\big) > v\}dv \le \int_0^{(T\,k\,H)^{\frac{1}{m}}\, h^{\frac{r-1}{m}}} dv +$$
$$+ \int_{(T\,k\,H)^{\frac{1}{m}}\, h^{\frac{r-1}{m}}}^{\infty} \frac{T\,k\,H\,h^{r-1}}{v^m}\, dv = \frac{m}{m-1}\, (T\,k\,H)^{\frac{1}{m}}\, h^{\frac{r-1}{m}} = B_1\, h^{\frac{r-1}{m}}\,,$$

where B_1 is given by (17). Inequality (16) is proved.

Finally, for a separable process $S(t)$ condition **D** implies continuity of the paths, see Gikhman and Skorokhod (1974). $\bigoplus$

Corollary. *Let condition* **D** *holds. Then for every* $0 < u < m,\ 0 < r' < r$

$$E(\sup_{0 \le t',\, t'' \le T} |\, S(t') - S(t'')\,|^u) \le \frac{m}{m-u}\, k_1\, H^{\frac{u}{m}}\, T^{\frac{ru}{m}}\,, \qquad (18)$$

where

$$k_1 = 2^u\,(1 - 2^{-\frac{r'-1}{m}})^{-u}\,(2^{r-r'} - 1)^{-\frac{u}{m}}\,.$$

Proof. Use (13) for the process $S(t)$. Let again $g(h) = h^{\frac{r'-1}{m}},\ \ 0 < r' < r$, and $q(C,h) = \dfrac{H}{C^m}\,h^{1+r-r'}$. Then according to (12)

$$G = T^{\frac{r'-1}{m}}\,(1 - 2^{-\frac{r'-1}{m}})^{-1}, \quad Q(C) = \frac{H}{C^m}\,T^{1+r-r'}\,(2^{r-r'} - 1)^{-1}.$$

Now,

$$Q(\delta/2G) = k_0\,\frac{H\,T^r}{\delta^m},$$

where

$$k_0 = 2^m\,(1 - 2^{-\frac{r'-1}{m}})^{-m}\,(2^{r-r'} - 1)^{-1}.$$

By (13) we have

$$P\{\sup_{0\le t',\,t''\le T} |\,S(t') - S(t'')\,| > \delta\} \le \frac{k_0\,H\,T^r}{\delta^m},$$

and

$$E(\sup_{0\le t',\,t''\le T} |\,S(t') - S(t'')\,|^u) = \int_0^\infty P\{\sup_{0\le t',\,t''\le T} |\,S(t') - S(t'')\,| > v^{\frac{1}{u}}\}dv$$

$$\le \int_0^A dv\, +\, \int_A^\infty \frac{k_0\,H\,T^r}{v^{\frac{m}{u}}}\,dv.$$

Choose A from the condition $k_0\,H\,T^r\,A^{-\frac{m}{u}} = 1$. After straightforward calculation we obtain

$$E(\sup_{0\le t',\,t''\le T} |\,S(t') - S(t'')\,|^u) \le \frac{m}{m-u}(k_0\,H\,T^r)^{\frac{u}{m}} = \frac{m}{m-u}\,k_1\,H^{\frac{u}{m}}\,T^{\frac{ru}{m}},$$

and (18) is proved. ⨁

Lemma 3, applied to the transformed pricing processes $S_g(t)$, yields the explicit solution in (10) and links parameters h and ε.

For example we get by substituting the corresponding upper bound in (10) the stronger inequality $B_1 h^{\frac{r-1}{m}} \le \varepsilon$, which guarantees that $h_\varepsilon = (\varepsilon/B_1)^{\frac{m}{r-1}}$ is the solution of (10). In sequel, if a partition Π_{N_ε} is chosen in such a way that the maximal step $d(\Pi_{N_\varepsilon}) \le (\varepsilon/B_1)^{\frac{m}{r-1}}$ then any a ε-optimal stopping moment τ'_ε in the class $\mathcal{M}_{\Pi_{N_\varepsilon},T}$ will be a 2ε-optimal stopping moment in the class $\mathcal{M}_{max,T}$.

Sometimes it is not convenient to apply Lemma 1 to the transformed pricing process $S_g(t)$ and it would be better to have similar estimates given in terms of increments of the pricing process S_t itself. Such estimates can be obtained in the case of smoothed pricing functions.

Let again $S(t), t \geq 0$ be a separable real-valued process for which the condition **D** holds. Let also $G(x,t)$ be a measurable real-valued function defined on $R \times R^+$. We consider the transformed process $S_G(t) = G(S(t),t)$, $t \geq 0$.

In the case of pricing processes transformation function is $G(x,t) = e^{-R_t} g(x,t)$ and the transformed pricing process is $S_g(t) = e^{-R_t} g(S_t,t), t \geq 0$.

In general case we do not make any assumptions about structure of transformation function $G(x,t)$. We assume only the following smoothness condition, which links the order of smoothness for function $G(x,t)$ with the parameter r in condition **D**:

E: (a) $G(x,t)$ is absolutely continuous upon x for every fixed $t \geq 0$ and upon t for every fixed $x \in R$; (b) for every $x \in R$ function $|\frac{\partial G(x,t)}{\partial t}| \leq K_1 |x|^{p_1}$ for almost all $t \in [0,T]$ with respect to Lebesgue measure, where $K_1 > 0$ and $0 \leq p_1 < r$; (c) for every $t \in [0,T]$ function $|\frac{\partial G(x,t)}{\partial x}| \leq K_2 |x|^{p_2}$ for almost all $x \in R$ with respect to Lebesgue measure, where $K_2 > 0$ and $0 \leq p_2 < r-1$.

Condition **E** guarantees the existence of the moments of the order m for increments of the process $S(t)$. Since nonlinear character of transformation function $G(x,t)$ we need also the following condition:

F: $E\,|S(0)|^m < \infty$.

In Lemma 3 an additional parameter $1 < r' < r$ was involved. Here we need to involve another additional parameter $1 < q < r'$. Let denote:

$$B_2 = \frac{m}{m-q}\, 2^{\frac{m+r-2q}{m}}\, (1 - 2^{-\frac{r'-q}{m}})^{-1}\, (1 - 2^{-\frac{r-r'}{q}})^{-\frac{q}{m}} \times \tag{19}$$

$$\times T^{\frac{q}{m}} \{ K_1^{\frac{m}{q}} M_1\,(p_1 m/q)\, T^{\frac{m-r}{q}} + K_2^{\frac{m}{q}}\, (M_2(p_2 m/(q-1)))^{\frac{q-1}{q}}\, H^{\frac{1}{q}} \}^{\frac{q}{m}},$$

where

$$M_1\,(u) = 2^{[u-1]_+} \cdot (E\,|S(0)|^u \,+\, H^{\frac{u}{m}}\, T^{\frac{ru}{m}}),$$

$$M_2\,(u) = 2^{[u-1]_+} \cdot \{E\,|S(0)|^u + \frac{m}{m-u} 2^u (1 - 2^{-\frac{r'-1}{m}})^{-u} (2^{r-r'}-1)^{-\frac{u}{m}} H^{\frac{u}{m}} T^{\frac{ru}{m}}\}.$$

Lemma 4. *Let conditions* **D** *with* $m \geq r > 1$ *and* **E**–**F** *hold. Then for every* q *and* r' *such that* $p_2 + 1 < q < r' < r$, $p_1 \leq q$ *and every* $0 < h \leq T$:

$$E\,\Delta_{h,T}(S_G(\cdot)) \leq B_2 h^{\frac{r-q}{m}}.$$

Proof. Assumption E implies

$$|G(S_t\,,t) - G(S_s\,,s)| \;\leq\; |G(S_t\,,t) - G(S_t\,,s)| \,+\, |G(S_t\,,s) - G(S_s\,,s)| \;\leq$$

$$\leq\; K_1\,|S_t|^{p_1}\,|t-s| \,+\, K_2 \sup_{t\,\in\,[0,T]} |S_t|^{p_2}\,|S_t \,-\, S_s|.$$

Let q and r' satisfy the conditions of Lemma 4. Then

$$E|\, G(S_t\,,t) - G(S_s\,,s)|^{\frac{m}{q}} \;\le 2^{\frac{m}{q}-1}\,\{K_1^{\frac{m}{q}}\;|t-s|^{\frac{m}{q}}\;E\,|S_t|^{\frac{mp_1}{q}}+$$

$$+K_2^{\frac{m}{q}}(E\,|S_t\; - \;S_s\,|^m)^{\frac{1}{q}}\;(E\sup_{t\in[0,T]}|S_t|^{\frac{p_2 m}{q-1}})^{\frac{q-1}{q}}\}. \tag{20}$$

Now, by **D** we have for $0\le\; u \le\; m$:

$$E|S_t|^u \;\le\; 2^{[u-1]_+}\times\{E|S(0)|^u \;+\; (E|\,S(t)-S(0)|^m)^{\frac{u}{m}}\}\;\le\; M_1\,(u), \tag{21}$$

and by **D** and Corollary for $0\le\; u <\; m$:

$$E\sup_{t\,\in\,[0,T]}|S_t|^u \;\le\; 2^{[u-1]_+}\times\{E\,\big|S(0)|^u \;+\; E\sup_{t\,\in\,[0,T]}|S(t)-S(0)\big|^u\}\;\le$$

$$\le\; 2^{[u-1]_+}\times\{E|S(0)|^u \;+\; \frac{m}{m-u}\,k_1\,H^{\frac{u}{m}}\,T^{\frac{ru}{m}}\}\;=\;M_2\,(u). \tag{22}$$

¿From (20) – (22) we obtain

$$E|G(S_t\,,t) - G(S_s\,,s)|^{\frac{m}{q}} \;\le\; H_1|t-s|^{\frac{r}{q}}, \tag{23}$$

where

$$H_1 = 2^{\frac{m}{q}-1}\{K_1^{\frac{m}{q}}\;T^{\frac{m-r}{q}}\;M_1(mp_1/q) \;+\; K_2^{\frac{m}{q}}\;H^{\frac{1}{q}}[M_2(mp_2/(q-1))]^{\frac{q-1}{q}}\}.$$

Finally, by Lemma 3 and inequality (23) we get

$$E\,\Delta_{h,T}(S_G(\cdot)) \le B_1\,(m/q,\,r/q,\,H_1)\;h^{(\frac{r}{q}-1)\,(\frac{m}{q})^{-1}} = B_1(m/q,\,r/q,\,H_1)\,h^{\frac{r-q}{m}}\,. \tag{24}$$

Here $B_1(m/q,\,r/q,\,H_1)$ is obtained from B_1, which is given in (17), by substitution m/q, r/q and H_1 instead of m, r and H, respectively; we substitute also in (17) r'/q instead of r'. We have

$$B_1(m/q,\,r/q,\,H_1)\;=\;\frac{m}{m-q}\,2^{\frac{r-q}{m}}(1-2^{-\frac{r'-q}{m}})^{-1}\times$$

$$\times(1-2^{-\frac{r-r'}{q}})^{-\frac{q}{m}}\;\times\;H_1^{\frac{q}{m}}\,T^{\frac{q}{m}}\;=\;B_2. \tag{25}$$

Now, (24) and (25) imply that

$$E\,\Delta_{h,T}\Big(S_G(\cdot)\Big)\;\le\;B_2\,h^{\frac{r-q}{m}}\,.$$

Lemma 4 is proved. ⨁

4. Skeleton approximations for the basic example

Let us illustrate the possible application of Lemmas 3 and 4 to the model where the pricing process S_t, $t \geq 0$ is given in the form:

$$S_t = S_0 \cdot \exp\{\int_0^t (a\,(u, I_u) - \frac{1}{2}\sigma(u)^2)\, du + \int_0^t \sigma(u)\, dw(u)\}, \quad t \geq 0,$$

where (a) $a(t,y)$ is a measurable real-valued functions defined on Z, (b) $\sigma(t) \geq 0$ is a measurable real-valued functions defined on R^+, (c) I_t, $t \geq 0$ is a measurable inhomogeneous in time Markov process, (d) $w(u)$, $u \geq 0$ is the Wiener process independent of process I_t, $t \geq 0$, (e) $Z_0 = (S_0, I_0)$ is a non-random value in Z.

In this case vector process $Z_t = (S_t, I_t)$, $t \geq 0$ is an inhomogeneous Markov process with the first component S_t, $t \geq 0$ is a continuous geometrical diffusion process controlled by process I_t, $t \geq 0$.

We assume the following condition:

$$\mathbf{G}\text{: (a) } A = \sup_{0 \leq t \leq T,\, y \in Y} |a(t,y) - \frac{1}{2}\,\sigma(t)^2| < \infty; \quad \text{(b) } B = \sup_{0 \leq t \leq T} \sigma(t) < \infty.$$

Lemma 5. *Let condition* **G** *holds. Then for any* $m > 2$

$$E\,|S_{t'} - S_{t''}|^m \leq H_m\,|t' - t''|^{m/2}, \quad 0 \leq t', \quad t'' \leq T,$$

where

$$H_m = \frac{1}{2}\,(2S_0\, e^{AT + \frac{1}{2} m B^2 T}\, T^{-\frac{1}{2}})^m\, ((e^{AT} - 1)^m + E\,|e^{BT^{\frac{1}{2}} N(0,1)} - 1|^m).$$

Proof. Fix $m > 2$ and denote $b(t,y) = a(t,y) - \frac{1}{2}\sigma(t)^2$, $0 \leq t \leq T, y \in Y$. We suppose that $S_0 > 0$. Then for every $t \in [0,T]$ $S_t > 0$ a.s. Fix $t \in [0,T]$ and positive h, such that $t + h \in [0,T]$. Consider the increment

$$|S_{t+h} - S_t| = S_t \cdot |\exp\{\int_t^{t+h} b(u, I_u)\, du + \int_t^{t+h} \sigma(u)\, dw(u)\} - 1| \leq$$

$$\leq S_t \cdot \{\exp(\int_t^{t+h} \sigma(u)\, dw(u)) \times (e^{Ah} - 1) + |\exp\Big(\int_t^{t+h} \sigma(u)\, dw(u)) - 1|\}.$$

Now, $S_t \leq S_0 e^{AT} \cdot \exp(\int_0^t \sigma(u)\, dw(u))$. Therefore

$$\frac{|S_{t+h} - S_t|}{S_0 e^{AT}} \leq \exp\{\int_0^{t+h} \sigma(u)\, dw(u)\} \times (e^{Ah} - 1)$$

$$+ \exp\{\int_0^t \sigma(u)\,dw(u)\} \times |\exp\{\int_t^{t+h} \sigma(u)\,dw(u)\} - 1|.$$

Then

$$E|\frac{S_{t+h} - S_t}{S_0 e^{AT}}|^m \le 2^{m-1}\,(e^{Ah} - 1)^m \cdot E\exp\{m \int_0^{t+h} \sigma(u)\,dw(u)\} +$$

$$+ 2^{m-1} \cdot E\exp\{m \int_0^t \sigma(u)\,dw(u)\} \cdot E|\exp\{\int_t^{t+h} \sigma(u)\,dw(u)\} - 1\,|^m. \quad (26)$$

For each $t \in [\,0, T\,]$ we have

$$E\exp\{m \int_0^t \sigma(u)\,dw(u)\} = \exp\{\frac{m^2}{2} \cdot \int_0^t \sigma^2(u)\,du\} \le e^{\frac{m^2 B^2 T}{2}}. \quad (27)$$

The inequality $|e^{\alpha z} - 1| \le |e^{\beta z} - 1|, \quad 0 \le \alpha \le \beta,\ z \in R$, implies

$$E|\exp\{\int_t^{t+h} \sigma(u)\,dw(u)\} - 1|^m = E|\exp\{(\int_t^{t+h} \sigma^2(u)\,du)^{\frac{1}{2}} \times N(0,1)\} - 1|^m \le$$

$$\le E|e^{B\sqrt{h}\cdot N(0,1)} - 1|^m \le (\sqrt{h/T})^m \times E|e^{B\sqrt{T}\cdot N(0,1)} - 1|^m. \quad (28)$$

Here we used the inequality

$$|e^{hz} - 1| \le \frac{h}{T}\,|e^{Tz} - 1|, \quad 0 < h \le T, \quad z \in R, \quad (29)$$

which follows from the convexity of the exponential function. From (26) – (29) we obtain finally

$$E|\frac{S_{t+h} - S_t}{S_0\,e^{AT}}|^m \le (\frac{h}{T})^{\frac{m}{2}} 2^{m-1} \times e^{\frac{1}{2}m^2 B^2 T}\{\,(e^{AT} - 1)^m + E\,|e^{B\sqrt{T}\cdot N(0,1)} - 1|^m\,\},$$

and

$$E|S_{t+h} - S_t|^m \le H_m\,h^{\frac{m}{2}}.$$

This completes the proof. ⨁

So, condition **D** holds and Lemma 4 can be applied to the pricing process $S_t, t \ge 0$ if condition **E** holds for the transformation function $G(x,t) = e^{-R_t} g(x,t)$.

Consider the case of standard American option. Here the transformation function

$$G(x,t) = e^{-\tilde{r}t}\,[x - K]_+,\ x \ge 0,\ 0 \le t \le T,$$

where $\tilde{r} > 0,\ K > 0$.

Let us apply Lemmas 4 and 5.

For $m > 2$ we have

$$E\,|S_{t'} - S_{t''}|^m \le H_m\,|t' - t''|^{m/2}, \quad 0 \le t', t'' \le T,$$

where H_m is given in Lemma 5. Thus **D** holds with $m > 2, r = \frac{m}{2}, H = H_m$.

Now, check the condition **E**. We have

$$|\frac{\partial G(x,t)}{\partial t}| \le \tilde{r},\ |\frac{\partial G(x,t)}{\partial x}| \le 1$$

(except the point $x = K$), therefore **E** holds with $p_1 = p_2 = 0,\ k_1 = \tilde{r},\ k_2 = 1$.

Fix q and r' such that $1 < q < r' < \frac{m}{2}$. By lemma 4 we have

$$E\,\Delta_{h,T}\,(S_G(\cdot)) \le B_2\,h^{\frac{1}{2}-\frac{q}{m}},$$

where B_2 is given by (19), with $r = \frac{m}{2},\ K_1 = \tilde{r},\ K_2 = 1,\ H = H_m,\ p_1 = p_2 = 0,\ M_1(0) = M_2(0) = 2$.

According to (9)

$$\Phi_g(\mathcal{M}_{max,T}) - \Phi_g(\mathcal{M}_{\Pi_N,T}) \le B_2\,d(\Pi_N)^{\frac{1}{2}-\frac{q}{m}} \le \varepsilon,$$

if $d(\Pi_N) \le (\varepsilon/B_2)^\alpha$, with $\alpha = (\frac{1}{2} - \frac{q}{m})^{-1}$.

To find $\Phi_g(\mathcal{M}_{\Pi_N,T})$ one can apply the results given papers Kukush and Silvestrov (2000a, 2000b). Let

$$\Pi_N = \{0 = t_0 < t_1 < ... < t_N = T\}\,.$$

In order to imbed the model in those considered in these papers one should consider the two component Markov chain $(S_n, I_n = (I'_n, I''_n))$, where

$$S_n = S_{t_n},\ I'_n = I_{t_n},\ I''_n = \exp\{\int\limits_{t_{n-1}}^{t_n} (a(u, I_u) - \frac{1}{2}\,\sigma(u)^2)du + \int\limits_{t_{n-1}}^{t_n} \sigma(u)dw(u)\}.$$

Let $r_k = \tilde{r}(t_{k+1} - t_k),\ k = 0, 1, ..., N-1,\ R_0 = 0,\ R_n = r_0 + r_1 + ... + r_{n-1},\ n = 1, 2, ..., N$.

The functional $\Phi_g(\tau)$ defined in (2) for $\tau \in \mathcal{M}_{\Pi_N,T}$ coincides with the functional

$$\Phi_g(\tau) = E\,e^{-R_\tau}\,[S_\tau - K]_+ \qquad (30)$$

introduced in Kukush and Silvestrov (2000a, 2000b) for the discrete Markov chain (S_n, I_n).

It follows from the formulas, which define Markov chain (S_n, I_n) that the first component can be given in the following dynamical form $S_n = S_{n-1} \cdot I''_n$. Also it is obvious that component I_n is also a Markov chain and it's transition probabilities depend only of the first component I'_n. That is why a conditions **A-C** used in Kukush and Silvestrov (2000a, 2000b) obviously hold. In particular the dynamical transition function $A(x, (y', y'')) = x \cdot y'$, which is derived from the formula $S_n = A(S_{n-1}, I_n) = S_{n-1} \cdot I''_n$, is convex and continuous in x for every (y', y'').

Assume additionally that

H: $D = \sup\limits_{0 \le t \le T,\ y \in Y} a(t, y) < \tilde{r}$.

Condition condition implies that condition **D**, introduced in Kukush and Silvestrov (2000a, 2000b), holds with $a_n \equiv 1$ (recall that we consider the case of standard American option). Really, for each $x > 0$

$$\frac{1}{x} E\{S_{t_{n+1}} / S_{t_n} = x, I_{t_n} = y\} =$$

$$E\{\exp\{\int\limits_{t_n}^{t_{n+1}} (a(u, I_u) - \frac{1}{2}\, \sigma(u)^2) du + \int\limits_{t_n}^{t_{n+1}} \sigma(u) dw(u)\} / I_{t_n} = y\} =$$

$$E\{\exp\{\int\limits_{t_n}^{t_{n+1}} a(u, I_u) du\} / I_{t_n} = y\} \le e^{D\,(t_{n+1} - t_n)} < e^{r_n}.$$

Therefore Theorem 2 from Kukush and Silvestrov (2000a, 2000b) is applicable now, and the structure of $\tau_{opt} \in \mathcal{M}_{\Pi_N, T}$ for the functional (30) is given in that theorem.

Remark also that if to replace **H** by

I: $E\{a(u, I_u) / I_t = y\} \ge \tilde{r}$, for each $0 \le t \le u \le T,\ y \in Y$,

then for $x > 0, t < s$:

$$\frac{1}{x}\, E\{S_{t_{n+1}} / S_{t_n} = x, I_{t_n} = y\} =$$

$$E\{\exp\{\int\limits_{t_n}^{t_{n+1}} (a(u, I_u) - \frac{1}{2}\, \sigma(u)^2) du + \int\limits_{t_n}^{t_{n+1}} \sigma(u) dw(u)\} / I_{t_n} = y\} =$$

$$= E\{\exp\{\int\limits_{t_n}^{t_{n+1}} a(u, I_u) du\} / I_{t_n} = y\} \ge e^{\tilde{r}\,(s-t)},$$

and the process $V_t = e^{-\tilde{r}t}[S_t - K]_+,\ 0 \le t \le T$ is a submartingale (compare with the proof of Theorem 4 from Kukush and Silvestrov (2000a)). Therefore under **I** for the functional (2) in the class $\mathcal{M}_{max,T}$ we have $\tau_{opt} = T$.

The cases of American type options with linear convex pay-off functions and with general convex pay-off functions can be considered by similar way with the use of corresponding results given in Kukush and Silvestrov (2000a, 2000b).

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