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Infill Asymptotics Inside Increasing Domains for the Least Squares Estimator in Linear Models

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Abstract. A linear model observed in a spatial domain is considered. Consistency and asymptotic normality of the least squares estimator is proved when the observations become dense in a sequence of increasing domains and the error terms are weakly dependent. Similar statements are obtained for the linear errors-in-variables model.

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1. Introduction

Analyzing statistical data one often encounters nonstandard problems. This paper deals with three problems: dependent observations, spatial data, and infill asymptotics inside increasing domain. Consistency and asymptotic normality of the least squares estimator is well-known when the observations are taken from a sequence of increasing domains. However, consistency is not valid in the case of infill asymptotics (see e.g. Lahiri [10], Fazekas et al. [6]). In this paper, we present conditions for consistency and asymptotic normality of the least squares estimator in a linear model when the observations become dense in a sequence of increasing domains.

Consider the linear model

$$z(\mathbf{x}) = \beta^\top f(\mathbf{x}) + \varepsilon(\mathbf{x}), \quad \mathbf{x} \in T_\infty \subset \mathbb{R}^d, \tag{1.1}$$

where d is a fixed positive integer, $z(\mathbf{x})$ is the observed random field, $\varepsilon(\mathbf{x})$ is the nonobserved random error term, $f(\mathbf{x})$ is the column vector of explanatory variables, f is a known function, and β is the unknown parameter to be estimated. We suppose that $\beta \in \mathbb{R}^p$, where p is a fixed positive integer, $f: T_\infty \to \mathbb{R}^p$ is continuous, $\{\varepsilon(\mathbf{x}),\ \mathbf{x} \in T_\infty\}$ is a mean zero random field.

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Describe the scheme of observations. Let $T_1, T_2, \dots,$ and T_∞ be domains in $\mathbb{R}^d$. Suppose that

$$T_1 \subset T_2 \subset T_3 \subset \dots, \quad \bigcup_{i=1}^{\infty} T_i = T_\infty. \tag{1.2}$$

Assume that T_i is compact for each i, T_∞ is of infinite Lebesgue measure. The nth set of observations consists of values of the random field $z(\mathbf{x})$ taken at points $\mathbf{x_k} \in T_n$, where $\mathbf{k} \in \mathcal{D}_n \subset \mathbb{Z}^d$. The choice of points $\mathbf{x_k}$ is the following. Divide $\mathbb{R}^d$ into hyperrectangles

$$\Delta_n(\mathbf{k}) = \prod_{j=1}^{d} \left(\frac{k_j}{N_{jn}}, \frac{k_j + 1}{N_{jn}} \right), \tag{1.3}$$

where $\mathbf{k} = (k_1, \dots, k_d)^\top \in \mathbb{Z}^d$ is a d-dimensional integer lattice point and $\{N_{jn}\}$ is an increasing and unbounded sequence of positive integers for each $j = 1, \dots, d$. Now, select the nth data sites $\mathbf{x_k}$, $\mathbf{k} \in \mathcal{D}_n$, by choosing an arbitrary point $\mathbf{x_k}$ from each $\Delta_n(\mathbf{k}) \cap T_n$ which is nonempty. Actually, each $\mathbf{x_k} = \mathbf{x_k}^{(n)}$ depends on n but to avoid complicated notation we omit superscript (n).

The least squares estimator of β is a measurable solution of the normal equation

$$D_n \widehat{\beta}_n = \sum_{\mathbf{k} \in \mathcal{D}_n} z(\mathbf{x_k}) f(\mathbf{x_k}), \tag{1.4}$$

where $D_n = \sum_{\mathbf{k} \in \mathcal{D}_n} f(\mathbf{x_k}) f(\mathbf{x_k})^\top$. If D_n is invertible then

$$\widehat{\beta}_n = D_n^{-1} \sum_{\mathbf{k} \in \mathcal{D}_n} z(\mathbf{x_k}) f(\mathbf{x_k}). \tag{1.5}$$

In Section 2 theorems on consistency and asymptotic normality of $\widehat{\beta}_n$ are given. In Section 3 similar statements are presented for the errors-in-variables model, i.e. for the model where $\mathbf{x_k}$'s are observed with error. In Section 4 appropriate versions of the Rosenthal inequality and the central limit theorem are proved for α-mixing random fields. Proofs of the asymptotic theorems are given in Section 5 and Section 6. An example is presented in Section 7. Simulation results are given in Section 8.

The following notation is used. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of all integers, $\mathbb{N}^d$ and $\mathbb{Z}^d$ are d-dimensional lattice points. $\mathbb{R}$ is the real line, $\mathbb{R}^d$ is the d-dimensional space with the usual Euclidean norm $\|\mathbf{x}\|$. In $\mathbb{R}^d$ we shall also consider the distance corresponding to the maximum norm

$$\varrho(\mathbf{x}, \mathbf{y}) = \max_{1 \leqslant i \leqslant d} |x_i - y_i|,$$

where $\mathbf{x} = (x_1, \dots, x_d)^\top$, $\mathbf{y} = (y_1, \dots, y_d)^\top$. The distance of two sets in $\mathbb{R}^d$ corresponding to the maximum norm is also denoted by ϱ.

Sign $^\top$ denotes the transpose of a matrix. Vectors $\mathbf{x}$, $\mathbf{k}$, $f(\mathbf{x}), \dots,$ are column vectors. The matrix norm we use is the operator norm in the Euclidean space. An eigenvalue of a matrix M is denoted by $\lambda(M)$, while the minimal eigenvalue by $\lambda_{\min}(M)$. Trace(M) is the trace of the matrix M. I_p denotes an identity matrix of type $p \times p$.

For real valued sequences $\{a_n\}$ and $\{b_n\}$, $a_n = o(b_n)$ (resp. $a_n = O(b_n)$) means that the sequence a_n/b_n converges to 0 (resp. is bounded).

We shall denote different constants with the same letter c. I$\{A\}$ denotes the indicator function of the set A. The Lebesgue measure of A is denoted by $\mu(A)$. $|\mathcal{D}|$ denotes the cardinality of the finite set $\mathcal{D}$.

We shall suppose the existence of an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $\omega \in \Omega$ denotes an elementary event. The σ-algebra generated by a set of events or by a set of random variables will be denoted by $\sigma\{.\}$. Sign $\mathbb{E}$ stands for the expectation. The variance and the covariance are denoted by var$(.)$ and cov$(.,.)$, respectively. The L_p-norm of a random (vector) variable η is

$$\|\eta\|_p = \left\{\mathbb{E}\|\eta\|^p\right\}^{1/p}, \quad 1 \leqslant p < \infty.$$

Sign $\Rightarrow$ denotes convergence in distribution. $\mathcal{N}(m, \Sigma)$ stands for the (vector) normal distribution with mean (vector) m and covariance (matrix) Σ.

2. Asymptotic Theorems for the Linear Model

To introduce assumptions we need some characteristics of f, $\varepsilon(\mathbf{x})$, and of the scheme of sampling. Let

$$M_n = \max_{\mathbf{x}\in T_n} \|f(\mathbf{x})\|, \qquad \omega_n(s) = \max_{\mathbf{x},\mathbf{y}\in T_n,\ \varrho(\mathbf{x},\mathbf{y}) \leqslant s} \|f(\mathbf{x}) - f(\mathbf{y})\|, \tag{2.1}$$

be the maximum and the modulus of continuity of f on T_n. Let δ_n denote the diameter of the hyperrectangle (1.3) and let ω_n denote the maximal variation of f on a hyperrectangle at the nth step:

$$\delta_n = \max_{1 \leqslant j \leqslant d} \frac{1}{N_{jn}}, \quad \omega_n = \omega_n(\delta_n). \tag{2.2}$$

Let

$$v_n = \left(\prod_{j=1}^{d} N_{jn}\right)^{-1} \tag{2.3}$$

be the volume (Lebesgue measure) of a hyperrectangle at the nth step. Let

$$T_n^\circ = \bigcup_{\mathbf{k}:\Delta_n(\mathbf{k})\subseteq T_n} \Delta_n(\mathbf{k}), \qquad \delta T_n = \bigcup_{\substack{\Delta_n(\mathbf{k})\cap T_n \neq \emptyset \\ \Delta_n(\mathbf{k})\cap T_n^c \neq \emptyset}} \Delta_n(\mathbf{k}), \tag{2.4}$$

be the 'interior' and the 'border' of T_n, respectively. (Here T_n^c denotes the complement of T_n.) We shall suppose that

$$\lim_{n\to\infty} \frac{\mu(\delta T_n)}{\mu(T_n)} = 0. \tag{2.5}$$

Introduce the following assumptions (where g_n denotes a positive real number).

$$\lambda_{\min}\left(\frac{1}{g_n\mu(T_n)}\int_{T_n} f(\mathbf{x})f(\mathbf{x})^\top \mathrm{d}\mathbf{x}\right) \geqslant \lambda_0 > 0 \quad \text{for each } n. \tag{2.6}$$

$$\lim_{n\to\infty} \frac{M_n^2\mu(\delta T_n)}{g_n\mu(T_n)} = 0. \tag{2.7}$$

$$\lim_{n\to\infty} \frac{M_n\omega_n}{g_n} = 0. \tag{2.8}$$

We need the notion of α-mixing (see e.g. Doukhan [2], Guyon [8]). Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras in $\mathcal{F}$. The α-mixing coefficient of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

$$\alpha(\mathcal{A},\mathcal{B}) = \sup\{|\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)| \ : \ A\in\mathcal{A},\, B\in\mathcal{B}\}.$$

The α-mixing coefficient of $\{\varepsilon(\mathbf{x})\colon\ \mathbf{x}\in T_\infty\}$ is

$$\alpha(r,u,v) = \sup\{\alpha(\mathcal{F}_{I_1},\mathcal{F}_{I_2})\colon\ \varrho(I_1,I_2)\geqslant r,\ |I_1|\leqslant u,\ |I_2|\leqslant v\},$$

where I_1 and I_2 are finite subsets in T_∞, $\mathcal{F}_{I_i} = \sigma\{\varepsilon(\mathbf{x})\colon\ \mathbf{x}\in I_i\}$, $i = 1, 2$. We shall use the following condition:

$$\int_0^\infty s^{d-1}\alpha^{\tau/(2+\tau)}(s,1,1)\,\mathrm{d}s < \infty, \quad \text{for some} \quad 0<\tau<1. \tag{2.9}$$

Let

$$\Lambda_n = \max_{1\leqslant j\leqslant d} N_{jn}, \qquad \lambda_n = \min_{1\leqslant j\leqslant d} N_{jn}. \tag{2.10}$$

Let $V_n = \sup_{\mathbf{x}\in T_n}\mathbb{E}(\varepsilon(\mathbf{x}))^2$. For $0<\tau<1$ consider the assumption

$$\lim_{n\to\infty} \frac{v_n^{1-\tau}M_n^{2-\tau}V_n^{(2-\tau)/2}\Lambda_n^d}{g_n^{2-\tau}\mu^{1-\tau}(T_n)} = 0. \tag{2.11}$$

Our consistency result is the following:

THEOREM 2.1. *Assume that there exist $\tau > 0$ and a sequence $\{g_n\}$ of positive numbers such that conditions (2.5), (2.6), (2.7), (2.8), (2.9), and (2.11) are satisfied. Then $\widehat{\beta}_n$ is consistent, that is*

$$\lim_{n\to\infty}\widehat{\beta}_n = \beta \tag{2.12}$$

in probability, where β is the true parameter.

Remark 2.1. Condition (2.5) enables us to formulate several assumptions in terms of integrals over T_n. It is motivated by the concept of Jordan measurability and it is mild. In most cases one can suppose that $\mu(\delta T_n) = 0$ (which implies (2.5) and also (2.7)). Moreover, instead of (2.5), condition $\mu(\delta T_n)/\mu(T_n)$ is bounded, is sufficient for Theorem 2.1. If we suppose that $\mathbb{E}|\varepsilon(\mathbf{x})|^{2+\tau}$ exists then condition (2.11) can be simplified. □

Remark 2.2. If M_n is bounded, then for (2.7) and (2.8) the following conditions are sufficient: $\lim_{n\to\infty} \mu(\delta T_n)/g_n\mu(T_n) = 0$ and functions $\{f(\mathbf{x})/g_n : \mathbf{x} \in T_n,\ n = 1, 2, \dots\}$ are equicontinuous.

Consider the case when M_n, V_n, and g_n are bounded. If $v_n \leqslant c\Lambda_n^{-d}$ (which is valid when the rectangles in (1.3) satisfy some regularity conditions) then (2.11) is satisfied if

$$\lim_{n\to\infty} \mu^{1-\tau}(T_n)v_n^{\tau} = \infty. \tag{2.13}$$

This condition means that the volume of T_n tends to ∞ rapidly enough compared to the rate of convergence as the volume of one hyperrectangle Δ_n tends to 0. □

Let

$$\Gamma(\mathbf{x}, \mathbf{y}) = \operatorname{var}(\varepsilon(\mathbf{x}), \varepsilon(\mathbf{y})), \quad \mathbf{x}, \mathbf{y} \in T_\infty, \tag{2.14}$$

be the covariance function of the random field $\varepsilon(\mathbf{x})$, and let

$$\omega_n^{\Gamma} = \sup_{\substack{\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}' \in T_n \\ \varrho(\mathbf{x},\mathbf{x}') \leqslant \delta_n,\ \varrho(\mathbf{y},\mathbf{y}') \leqslant \delta_n}} \|\Gamma(\mathbf{x}, \mathbf{y}) - \Gamma(\mathbf{x}', \mathbf{y}')\| \tag{2.15}$$

be the maximal variation of $\Gamma(\mathbf{x}, \mathbf{y})$ on a hyperrectangle Δ_n. Let

$$r_n = \left\| v_n D_n - \int_{T_n} f(\mathbf{x}) f(\mathbf{x})^{\top} \mathrm{d}\mathbf{x} \right\|. \tag{2.16}$$

Our asymptotic normality result is the following:

THEOREM 2.2. *Suppose that (2.5) and the following conditions are satisfied.*

$$\int_0^{\infty} s^{d-1}\alpha(s, i, j)\,\mathrm{d}s < \infty, \quad \textit{for} \quad i + j \leqslant 4. \tag{2.17}$$

$$\alpha(s, 1, \infty) = o(s^{-d}), \quad \textit{as} \quad s \to \infty. \tag{2.18}$$

There exists a $\tau > 0$ *such that (2.9) is satisfied and there exists a sequence* $\{h_n\}$ *of positive numbers such that*

$$\{\left(h_n^{-1}\|f(\mathbf{x})\|\cdot|\varepsilon(\mathbf{x})|\right)^{2+\tau} : \mathbf{x} \in T_n,\ n = 1, 2, \dots\} \quad \textit{are uniformly integrable} \tag{2.19}$$

with respect to the underlying probability measure.

$$\liminf_{n\to\infty} \frac{1}{h_n^2 \mu(T_n)} \lambda_{\min} \left(\int_{T_n} \int_{T_n} \Gamma(\mathbf{x},\mathbf{y}) f(\mathbf{x}) f(\mathbf{y})^\top \mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \right) > 0. \tag{2.20}$$

$$\Lambda_n = O(\lambda_n), \quad \text{as} \quad n \to \infty. \tag{2.21}$$

$$\lim_{n\to\infty} h_n^{-2} (M_n^2 \omega_n^\Gamma + V_n M_n \omega_n) = 0. \tag{2.22}$$

Then

$$\left(\int_{T_n} \int_{T_n} \Gamma(\mathbf{x},\mathbf{y}) f(\mathbf{x}) f(\mathbf{y})^\top \,\mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \right)^{-1/2} \left(v_n \sum_{\mathbf{k}\in\mathcal{D}_n} f(\mathbf{x_k}) f(\mathbf{x_k})^\top \right) (\widehat{\beta}_n - \beta) \Rightarrow \mathcal{N}(0, I_p), \quad \textit{as } n \to \infty. \tag{2.23}$$

If additionally, there exists a sequence $\{g_n\}$ *of positive numbers such that conditions (2.6), (2.7), (2.8), (2.11) and*

$$\lim_{n\to\infty} \frac{r_n}{g_n \mu(T_n)} = 0 \tag{2.24}$$

are satisfied, then

$$\left(\int_{T_n} \int_{T_n} \Gamma(\mathbf{x},\mathbf{y}) f(\mathbf{x}) f(\mathbf{y})^\top \,\mathrm{d}\mathbf{x}\, \mathrm{d}\mathbf{y} \right)^{-1/2} \left(\int_{T_n} f(\mathbf{x}) f(\mathbf{x})^\top \,\mathrm{d}\mathbf{x} \right) (\widehat{\beta}_n - \beta) \Rightarrow \mathcal{N}(0, I_p)\,, \quad \textit{as } n \to \infty. \tag{2.25}$$

3. Asymptotic Theorems for the Linear Errors-in-Variables Model

In this section we shall consider the linear errors-in-variables model. The observed random field and the scheme of observation are the same as in the case of the linear model. The only difference is that we observe $\mathbf{x_k}$ with error. The nth set of observation is the following:

$$z(\mathbf{x_k}) = \beta^\top f(\mathbf{x_k}) + \varepsilon(\mathbf{x_k}), \quad \mathbf{k} \in \mathcal{D}_n, \tag{3.1}$$

$$\mathbf{u_k} = \mathbf{x_k} + \vartheta_\mathbf{k}, \quad \mathbf{k} \in \mathcal{D}_n. \tag{3.2}$$

This model means that we observe $z(\mathbf{x_k})$ but we do not observe $\mathbf{x_k}$. Instead of $\mathbf{x_k}$ we observe $\mathbf{u_k}$; $\mathbf{x_k}$ is considered to be non-random that is we deal with a functional errors-in-variables model (see Fuller [7]). We suppose that in the case of nth observation

$$\mathbf{u_k} \in \Delta_n(\mathbf{k}) \quad \text{for each} \quad \mathbf{k} \in \mathcal{D}_n. \tag{3.3}$$

It means that we can identify the hyperrectangle $\Delta_n(\mathbf{k})$ in which we have the observation $z(\mathbf{x_k})$ but we do not know precisely in which point $\mathbf{x_k}$ inside the hyperrectangle was the random field observed.

As tools and results for the errors-in-variables model are similar to those for the linear model we shall denote by tilde that a quantity corresponds to the errors-in-variables model.

The so called naive least squares estimator of β is the minimum point of

$$Q(\beta) = \frac{1}{|\mathcal{D}_n|} \sum_{\mathbf{k}\in\mathcal{D}_n} \left[z(\mathbf{x_k}) - \beta^\top f(\mathbf{u_k})\right]^2, \quad \beta \in \mathbb{R}^p. \tag{3.4}$$

It is known that there is a random variable $\tilde{\beta}_n$ which is the minimum of $Q(\beta)$. $\tilde{\beta}_n$ is the solution of the normal equation:

$$\tilde{D}_n \tilde{\beta}_n = \sum_{\mathbf{k}\in\mathcal{D}_n} z(\mathbf{x_k}) f(\mathbf{u_k}), \tag{3.5}$$

where $\tilde{D}_n = \sum_{\mathbf{k}\in\mathcal{D}_n} f(\mathbf{u_k}) f(\mathbf{u_k})^\top$. If $\tilde{D}_n$ is invertible then

$$\tilde{\beta}_n = \tilde{D}_n^{-1} \sum_{\mathbf{k}\in\mathcal{D}_n} z(\mathbf{x_k}) f(\mathbf{u_k}). \tag{3.6}$$

The following theorem shows consistency of $\tilde{\beta}_n$. The conditions are the same as the ones for linear models in Theorem 2.1.

THEOREM 3.1. *Consider the model (3.1), (3.2), (3.3). Assume that there exist* $\tau > 0$ *and a sequence* $\{g_n\}$ *of positive numbers such that conditions (2.5), (2.6), (2.7), (2.8), (2.9) and (2.11) are satisfied. Then* $\tilde{\beta}_n$ *is consistent, that is*

$$\lim_{n\to\infty} \tilde{\beta}_n = \beta \tag{3.7}$$

in probability, where β *is the true parameter.*

Instead of (2.5), condition $\mu(\delta T_n)/\mu(T_n)$ is bounded, is sufficient for Theorem 3.1.

We remark that for general nonlinear errors-in-variables models the naive estimator is not consistent therefore we do not deal with that case (see Fazekas and Kukush [3] for an alternative estimator that is consitent).

The following theorem shows asymptotic normality of $\tilde{\beta}_n$. For a fixed n and a fixed $\mathbf{x} \in T_n$ define $M_n(\mathbf{x})$ as the supremum of $\|f(\mathbf{x})\|$ on the hyperrectangle $\Delta_n(\mathbf{k})$ that contains $\mathbf{x}$. Let

$$\tilde{r}_n = \left\| v_n \tilde{D}_n - \int_{T_n} f(\mathbf{x}) f(\mathbf{x})^\top \,\mathrm{d}\mathbf{x} \right\|. \tag{3.8}$$

THEOREM 3.2. *Consider the model (3.1), (3.2), (3.3). Suppose that (2.5), (2.17), (2.18), (2.21) and the following conditions are satisfied. There exists a* $\tau > 0$ *such* *that (2.9) is satisfied and there exists a sequence* $\{h_n\}$ *of positive numbers such that (2.20), (2.22) are satisfied. Moreover*

$$\{\left(h_n^{-1}M_n(\mathbf{x})|\varepsilon(\mathbf{x})|\right)^{2+\tau}: \ \mathbf{x}\in T_n,\ n=1,2,\dots\} \text{ are uniformly integrable} \tag{3.9}$$

with respect to the underlying probability measure.

$$\lim_{n\to\infty}\frac{\sqrt{\mu(T_n)}M_n\omega_n}{h_n}=0. \tag{3.10}$$

Then

$$\left(\int_{T_n}\int_{T_n}\Gamma(\mathbf{x},\mathbf{y})f(\mathbf{x})f(\mathbf{y})^\top\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right)^{-1/2}\left(\upsilon_n\sum_{\mathbf{k}\in\mathcal{D}_n}f(\mathbf{u_k})f(\mathbf{u_k})^\top\right)(\tilde{\beta}_n-\beta)$$
$$\Rightarrow\mathcal{N}(0,I_p),\quad \text{as } n\to\infty. \tag{3.11}$$

If additionally, there exists a sequence $\{g_n\}$ *of positive numbers such that conditions (2.6), (2.7), (2.8), (2.11) and*

$$\lim_{n\to\infty}\frac{\tilde{r}_n}{g_n\mu(T_n)}=0, \tag{3.12}$$

$$\lim_{n\to\infty}\frac{\tilde{r}_nM_n\omega_n}{h_ng_n\sqrt{\mu(T_n)}}=0 \tag{3.13}$$

are satisfied, then

$$\left(\int_{T_n}\int_{T_n}\Gamma(\mathbf{x},\mathbf{y})f(\mathbf{x})f(\mathbf{y})^\top\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}\right)^{-1/2}\left(\int_{T_n}f(\mathbf{x})f(\mathbf{x})^\top\,\mathrm{d}\mathbf{x}\right)(\tilde{\beta}_n-\beta)$$
$$\Rightarrow\mathcal{N}(0,I_p),\quad \text{as } n\to\infty. \tag{3.14}$$

4. Properties of Mixing Fields

In this part we adapt some theorems of discrete parameter random field to the case of our sampling scheme. We concentrate on the Rosenthal inequality and the central limit theorem.

The α-mixing coefficient of the random field $\{\varepsilon(\mathbf{x}):\ \mathbf{x}\in T_\infty\}$ is denoted by $\alpha(r,i,j)$.

Define the discrete parameter random field $Y_n(\mathbf{k})$ as follows. For each $n=1,2,\dots$, and for each $\mathbf{k}\in\mathcal{D}_n$

$$\text{let } Y_n(\mathbf{k}) \text{ be a Borel measurable function of } \varepsilon(\mathbf{x}_\mathbf{k}^{(n)}). \tag{4.1}$$

Then $Y_n(\mathbf{k})$ is a discrete parameter random field and we can apply results concerning random fields with parameter in $\mathbb{Z}^d$. Let $\alpha_n(r, i, j)$ denote the α-mixing coefficients of $Y_n(\mathbf{k})$.

As $\mathbf{x_k} \in \Delta_n(\mathbf{k})$ and $\mathbf{x_l} \in \Delta_n(\mathbf{l})$ (where $\mathbf{l} = (l_1, \dots, l_d)^\top$) we have

$$\varrho(\mathbf{x_k}, \mathbf{x_l}) \geqslant \max\left\{\frac{|k_1 - l_1| - 1}{N_{1n}}, \dots, \frac{|k_d - l_d| - 1}{N_{dn}}\right\} \geqslant \frac{\varrho(\mathbf{k}, \mathbf{l}) - 1}{\Lambda_n}, \tag{4.2}$$

and

$$\varrho(\mathbf{x_k}, \mathbf{x_l}) \leqslant \frac{\varrho(\mathbf{k}, \mathbf{l}) + 1}{\lambda_n}, \tag{4.3}$$

where Λ_n, λ_n are given in (2.10). Therefore the α-mixing coefficients $\alpha_n(r, i, j)$ of $Y_n(\mathbf{k})$ satisfy

$$\alpha\left(\frac{r+1}{\lambda_n}, i, j\right) \leqslant \alpha_n(r, i, j) \leqslant \alpha\left(\frac{r-1}{\Lambda_n}, i, j\right), \quad r = 1, 2, \dots. \tag{4.4}$$

LEMMA 4.1. *For $\gamma > 0$ and positive integers i, j, n*

$$\sum_{r=1}^{\infty} r^{d-1} \alpha_n^{\gamma}(r, i, j) \leqslant c\left(1 + \Lambda_n^d \int_0^{\infty} r^{d-1} \alpha^{\gamma}(r, i, j)\, \mathrm{d}r\right), \tag{4.5}$$

where the constant c depends only on d.

The following covariance inequalities are basic tools for mixing fields.

Remark 4.1. (See Doukhan [2], p. 9.)

$$|\operatorname{cov}(X, Y)| \leqslant 8[\alpha(\sigma(X), \sigma(Y))]^{1/r} \|X\|_p \|Y\|_q,$$
$$\text{for } r, p, q \geqslant 1, \quad \frac{1}{r} + \frac{1}{p} + \frac{1}{q} = 1. \tag{4.6}$$

In the special case when X and Y belong to L_∞

$$|\operatorname{cov}(X, Y)| \leqslant 4[\alpha(\sigma(X), \sigma(Y))] \|X\|_\infty \|Y\|_\infty. \tag{4.7}$$

The inequality below is a special case of the Rosenthal inequality.

LEMMA 4.2. *Let $Y_\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^d$, be centered random variables with $\mathbb{E}|Y_\mathbf{k}|^{l+\tau} < \infty$, $\mathbf{k} \in \mathbb{Z}^d$. Introduce the following notation:*

$$L(h, \tau, \mathcal{D}) = \sum\nolimits_{\mathbf{k} \in \mathcal{D}} (\mathbb{E}|Y_\mathbf{k}|^{h+\tau})^{h/(h+\tau)}, \tag{4.8}$$

if $1 < h \leqslant 2$, $\tau \geqslant 0$ and $\mathcal{D}$ is a finite set in $\mathbb{Z}^d$. Let

$$c_{1,1}^{(\tau)} = 1 + \sum_{s=1}^{\infty} s^{d-1} [\alpha_Y(s, 1, 1)]^{\tau/(2+\tau)}, \tag{4.9}$$

where $\alpha_Y(s, 1, 1)$ *is the* α*-mixing coefficient of* $\{Y_{\mathbf{k}}\}$*. Now, let* $1 < l \leqslant 2$ *and* $\tau > 0$*. Assume that* $c_{1,1}^{(\tau)} < \infty$*. Then there is a constant* c *such that*

$$\mathbb{E}\left|\sum\nolimits_{\mathbf{k}\in\mathcal{D}} Y_{\mathbf{k}}\right|^{l} \leqslant c\cdot c_{1,1}^{(\tau)} L(l, \tau, \mathcal{D}), \tag{4.10}$$

for any finite subset $\mathcal{D}$ *of* $\mathbb{Z}^d$.

The proof of the Rosenthal inequality (4.10) follows from (4.12) below, by using the so called interpolation lemma. Details and the general form of the Rosenthal inequality can be found e.g. in Fazekas et al. [5]. It is easy to see that the Rosenthal inequality is valid for vector valued random variables $Y_{\mathbf{k}}$, too. In that case (4.10) is of the form

$$\mathbb{E}\left\|\sum\nolimits_{\mathbf{k}\in\mathcal{D}} Y_{\mathbf{k}}\right\|^{l} \leqslant c\cdot c_{1,1}^{(\tau)} \sum\nolimits_{\mathbf{k}\in\mathcal{D}} \left(\mathbb{E}\|Y_{\mathbf{k}}\|^{l+\tau}\right)^{l/(l+\tau)}. \tag{4.11}$$

Remark 4.2. Using the notation of Lemma 4.2, let $\tau > 0$, and assume that $c_{1,1}^{(\tau)} < \infty$. Then there is a constant c such that

$$\sum\nolimits_{\mathbf{k},\mathbf{l}\in\mathcal{D}} |\operatorname{cov}(Y_{\mathbf{k}}, Y_{\mathbf{l}})| \leqslant c\cdot c_{1,1}^{(\tau)} L(2, \tau, \mathcal{D}), \tag{4.12}$$

for any finite subset $\mathcal{D}$ of $\mathbb{Z}^d$. □

Now, we turn to the version of the central limit theorem appropriate to our sampling scheme. Our theorem is a modification of Theorem 3.3.1 in Guyon [8] (the proof is also an appropriate version of the one presented in Guyon [8], for details see Fazekas and Kukush [4]). However, our conditions are stronger than those of Guyon [8] as we need uniform integrability. The reason is that one step in Guyon's proof is clear for us only if uniform integrability is supposed (see Bolthausen [1] for the stationary case when uniform integrability is not needed).

We concentrate on the case when $\varepsilon(\mathbf{x})$ and $\varepsilon(\mathbf{y})$ are dependent if $\mathbf{x}$ and $\mathbf{y}$ are close to each other. Therefore, our theorem does not cover the case when $Y_n(\mathbf{k})$'s are independent and identically distributed. On the other hand if $\varepsilon(\mathbf{x})$ is a stationary field with continuous covariance function then the covariance is close to a fixed positive number inside a small hyperrectangle. We intend to cover this case. Remind that $\mathcal{D}_n$ is a sequence of finite sets in $\mathbb{Z}^d$ with $\lim_{n\to\infty} |\mathcal{D}_n| = \infty$.

THEOREM 4.1. *Let* $\varepsilon(\mathbf{x})$ *be a random field and let* $Y_n(\mathbf{k})$ *be defined by (4.1) and suppose that* $\mathbb{E}Y_n(\mathbf{k}) = 0$ *for* $\mathbf{k} \in \mathcal{D}_n$, $n = 1, 2, \ldots$*. Let* $S_n = \sum_{\mathbf{k}\in\mathcal{D}_n} Y_n(\mathbf{k})$, $n = 1, 2, \ldots$, $\sigma_n^2 = \operatorname{var}(S_n)$*. Suppose that there exists a* $\tau > 0$ *such that (2.9) is satisfied and*

$$\{|Y_n(\mathbf{k})|^{2+\tau} : \mathbf{k} \in \mathcal{D}_n, n = 1, 2, \ldots\} \text{ are uniformly integrable}. \tag{4.13}$$

Then

$$\limsup_{n\to\infty} \frac{1}{\Lambda_n^d |\mathcal{D}_n|} \sum\nolimits_{\mathbf{k},\mathbf{l}\in\mathcal{D}_n} |\operatorname{cov}(Y_n(\mathbf{k}), Y_n(\mathbf{l}))| < \infty. \tag{4.14}$$

If additionally, conditions (2.17), (2.18), (2.21), and

$$\liminf_{n\to\infty} \frac{\sigma_n^2}{\Lambda_n^d |\mathcal{D}_n|} > 0 \tag{4.15}$$

are satisfied, then $\sigma_n^{-1} S_n \Rightarrow \mathcal{N}(0,1)$, *as* $n \to \infty$.

Theorem 4.1 is valid for vector valued random fields, too.

Remark 4.3. Let $\varepsilon(\mathbf{x})$ be a random field. For each $n = 1, 2, \ldots$, and for each $\mathbf{k} \in \mathcal{D}_n$ let $Y_n(\mathbf{k})$ be a centered p-dimensional random vector that is $\varepsilon(\mathbf{x}_{\mathbf{k}}^{(n)})$-measurable. Let $S_n = \sum_{\mathbf{k}\in\mathcal{D}_n} Y_n(\mathbf{k})$, $n = 1, 2, \ldots$, $\Sigma_n = \mathrm{var}(S_n)$. Assume that conditions (2.17), (2.18), and (2.21) are satisfied. Moreover, assume that there exists a $\tau > 0$ such that (2.9) is satisfied, and

$$\{\|Y_n(\mathbf{k})\|^{2+\tau} \;:\; \mathbf{k} \in \mathcal{D}_n, n = 1, 2, \ldots\} \text{ are uniformly integrable}. \tag{4.16}$$

Assume that

$$\liminf_{n\to\infty} \lambda_{\min}\left(\Lambda_n^{-d} |\mathcal{D}_n|^{-1} \Sigma_n\right) > 0. \tag{4.17}$$

Then $\Sigma_n^{-1/2} S_n \Rightarrow \mathcal{N}(0, I_p)$, as $n \to \infty$. □

5. Proofs for the Linear Model

According to (1.1) and (1.5)

$$\widehat{\beta}_n - \beta = \left(\frac{v_n}{g_n \mu(T_n)} D_n\right)^{-1} K_n, \tag{5.1}$$

where

$$K_n = \frac{v_n}{g_n \mu(T_n)} \sum_{\mathbf{k}\in\mathcal{D}_n} \varepsilon(\mathbf{x}_{\mathbf{k}}) f(\mathbf{x}_{\mathbf{k}}). \tag{5.2}$$

Remind that sets T_n° and δT_n are defined in (2.4).

LEMMA 5.1. *Assume that (2.6), (2.7), and (2.8) are satisfied. Then*

$$\liminf_{n\to\infty} \lambda_{\min}\left(\frac{v_n}{g_n \mu(T_n)} D_n\right) > 0. \tag{5.3}$$

Proof.

$$\begin{aligned} r_n &= \left\| v_n D_n - \int_{T_n} f(\mathbf{x}) f(\mathbf{x})^\top \,\mathrm{d}\mathbf{x} \right\| \leqslant \int_{T_n - T_n^\circ} \| f(\mathbf{x}) f(\mathbf{x})^\top \| \,\mathrm{d}\mathbf{x} + \\ &\quad + v_n \sum_{\{\mathbf{k}:\ \Delta_n(\mathbf{k}) \subseteq T_n^\circ\}} \sup_{\mathbf{x} \in \Delta_n(\mathbf{k})} \| f(\mathbf{x_k}) f(\mathbf{x_k})^\top - f(\mathbf{x}) f(\mathbf{x})^\top \| + \\ &\quad + v_n \sum_{\{\mathbf{k}\,:\,\mathbf{x_k} \in \delta T_n\}} \| f(\mathbf{x_k}) f(\mathbf{x_k})^\top \| \\ &\leqslant 2c\mu(\delta T_n) M_n^2 + 2c\mu(T_n^\circ) M_n \omega_n . \end{aligned}$$

Therefore, by (2.7) and (2.8),

$$\frac{r_n}{g_n \mu(T_n)} \leqslant \frac{2c}{g_n} \left(\frac{\mu(\delta T_n)}{\mu(T_n)} M_n^2 + \frac{\mu(T_n^\circ)}{\mu(T_n)} M_n \omega_n \right) \to 0, \tag{5.4}$$

as $n \to \infty$. Now (2.6) implies the result. □

Remark 5.1. 1. In the last step of the above lemma we used the following. Let A and B be symmetric matrices. Then

$$\lambda_{\min}(A) \geqslant \lambda_{\min}(B) - \|A - B\|.$$

To see this fix an arbitrary vector x with $\|x\| = 1$. Then $x^\top B x \geqslant \lambda_{\min}(B)$. Moreover,

$$x^\top A x = x^\top B x + x^\top (A - B) x \geqslant \lambda_{\min}(B) - \|A - B\|,$$

which gives the above inequality. We applied this result with $A = (g_n \mu(T_n))^{-1} v_n D_n$ and $B = (g_n \mu(T_n))^{-1} \int_{T_n} f(\mathbf{x}) f(\mathbf{x})^\top \,\mathrm{d}\mathbf{x}$.

2. The above lemma implies that under conditions (2.6), (2.7), and (2.8), D_n is nonsingular for large n's. □

LEMMA 5.2. *If (2.5), (2.7), (2.9) and (2.11) are satisfied then* $\lim_{n\to\infty} K_n = 0$ *in probability.*

Proof. By the Rosenthal inequality (i.e. Lemma 4.2) and by Lemma 4.1, for $0 < \tau < 1$,

$$\begin{aligned} \mathbb{E}\|K_n\|^{2-\tau} &\leqslant \frac{c v_n^{2-\tau}}{g_n^{2-\tau} \mu^{2-\tau}(T_n)} \times \\ &\quad \times \left(1 + \sum_{s=1}^{\infty} s^{d-1} \alpha_n^{\tau/(2+\tau)}(s, 1, 1) \right) \sum_{\mathbf{k} \in \mathcal{D}_n} \left(\mathbb{E}\|\varepsilon(\mathbf{x_k}) f(\mathbf{x_k})\|^2 \right)^{(2-\tau)/2} \\ &\leqslant \frac{c v_n^{2-\tau} |\mathcal{D}_n| M_n^{2-\tau} V_n^{(2-\tau)/2}}{g_n^{2-\tau} \mu^{2-\tau}(T_n)} \left(1 + \Lambda_n^d \int_0^\infty s^{d-1} \alpha^{\tau/(2+\tau)}(s, 1, 1) \,\mathrm{d}s \right), \end{aligned}$$

where $V_n = \sup_{\mathbf{x} \in T_n} \mathbb{E}(\varepsilon(\mathbf{x}))^2$.

As $v_n|\mathcal{D}_n|$ is the full volume $\mu(T_n \cup \delta T_n)$, and by (2.5), $\mu(\delta T_n)/\mu(T_n)$ is bounded, so $v_n|\mathcal{D}_n| \leqslant c\mu(T_n)$. Therefore, using also (2.9), we obtain

$$\mathbb{E}\|K_n\|^{2-\tau} \leqslant \frac{cv_n^{1-\tau} M_n^{2-\tau} V_n^{(2-\tau)/2}(\Lambda_n^d + 1)}{g_n^{2-\tau}\mu^{1-\tau}(T_n)}.$$

According to (2.11), this expression tends to 0. The lemma is proved. □

Proof of Theorem 2.1. It is a simple consequence of Lemma 5.1 and Lemma 5.2. □

Proof of Theorem 2.2 By (1.4), we have

$$h_n^{-1} D_n(\widehat{\beta}_n - \beta) = h_n^{-1} \sum_{\mathbf{k}\in\mathcal{D}_n} \varepsilon(\mathbf{x_k}) f(\mathbf{x_k}). \tag{5.5}$$

Let

$$\begin{aligned} h_n^{-2}\Sigma_n &= \operatorname{var}\left(h_n^{-1} \sum_{\mathbf{k}\in\mathcal{D}_n} \varepsilon(\mathbf{x_k}) f(\mathbf{x_k}) \right) \\ &= h_n^{-2} \sum_{\mathbf{k},\mathbf{l}\in\mathcal{D}_n} \Gamma(\mathbf{x_k}, \mathbf{x_l}) f(\mathbf{x_k}) f(\mathbf{x_l})^\top \end{aligned} \tag{5.6}$$

be the covariance matrix of our sum. We have to prove (4.17). To this end it is sufficient to prove that

$$\liminf_{n\to\infty} \lambda_{\min}\left(h_n^{-2}(\mu(T_n))^{-1} v_n^2 \Sigma_n\right) > 0. \tag{5.7}$$

Let

$$\Sigma_n^c = \int_{T_n}\int_{T_n} \Gamma(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) f(\mathbf{y})^\top \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y}. \tag{5.8}$$

We shall prove that

$$\pi_n = (h_n^2 \mu(T_n))^{-1} \left\| v_n^2 \Sigma_n - \Sigma_n^c \right\| \to 0, \tag{5.9}$$

as $n \to \infty$. By (2.19), $h_n^{-2}|\Gamma(\mathbf{x}, \mathbf{y})|\|f(\mathbf{x})\|\|f(\mathbf{y})\|$ is bounded, therefore (by (2.5)) it is sufficient to consider the case when T_n is a union of the hyperrectangles $\Delta_n(\mathbf{k})$, $\mathbf{k} \in \mathcal{D}_n$. Then

$$\pi_n \leqslant \frac{1}{h_n^2 \mu(T_n)} \sum_{\mathbf{k},\mathbf{l}\in\mathcal{D}_n} c_{\mathbf{kl}},$$

where

$$c_{\mathbf{kl}} = \left\| \int_{\Delta_n(\mathbf{k})}\int_{\Delta_n(\mathbf{l})} \Gamma(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) f(\mathbf{y})^\top \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} - v_n^2 \Gamma(\mathbf{x_k}, \mathbf{x_l}) f(\mathbf{x_k}) f(\mathbf{x_l})^\top \right\|.$$

It is easy to see that

$$\begin{aligned} c_{\mathbf{kl}} &\leqslant cv_n^2 \sup_{\substack{\mathbf{x},\mathbf{y},\mathbf{x}',\mathbf{y}'\in T_n \\ \varrho(\mathbf{x},\mathbf{x}')\leqslant\delta_n,\ \varrho(\mathbf{y},\mathbf{y}')\leqslant\delta_n}} \|\Gamma(\mathbf{x},\mathbf{y})f(\mathbf{x})f(\mathbf{y})^\top - \Gamma(\mathbf{x}',\mathbf{y}')f(\mathbf{x}')f(\mathbf{y}')^\top\| \\ &\leqslant cv_n^2(M_n^2\omega_n^\Gamma + V_n M_n \omega_n). \end{aligned} \tag{5.10}$$

Moreover, by the covariance inequality,

$$|\Gamma(\mathbf{x},\mathbf{y})| \leqslant c\alpha^{\tau/(2+\tau)}(\varrho_{\mathbf{k},\mathbf{l}}, 1, 1)\|\varepsilon(\mathbf{x})\|_{2+\tau}\|\varepsilon(\mathbf{y})\|_{2+\tau},$$

where $\varrho_{\mathbf{k},\mathbf{l}} = \varrho\left(\Delta_n(\mathbf{k}), \Delta_n(\mathbf{l})\right)$. As $\mathbb{E}|\varepsilon(\mathbf{x})|^{2+\tau}\|f(\mathbf{x})\|^{2+\tau}h_n^{-(2+\tau)}$ is bounded,

$$\left\| \int_{\Delta_n(\mathbf{k})}\int_{\Delta_n(\mathbf{l})} \Gamma(\mathbf{x},\mathbf{y})f(\mathbf{x})f(\mathbf{y})^\top \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} \right\| \leqslant cv_n^2h_n^2\alpha^{\tau/(2+\tau)}(\varrho_{\mathbf{k},\mathbf{l}}, 1, 1). \tag{5.11}$$

In the same way

$$v_n^2\|\Gamma(\mathbf{x_k},\mathbf{x_l})f(\mathbf{x_k})f(\mathbf{x_l})^\top\| \leqslant cv_n^2h_n^2\alpha^{\tau/(2+\tau)}(\varrho_{\mathbf{k},\mathbf{l}}, 1, 1). \tag{5.12}$$

Therefore

$$c_{\mathbf{kl}} \leqslant cv_n^2h_n^2\alpha^{\tau/(2+\tau)}(\varrho_{\mathbf{k},\mathbf{l}}, 1, 1) \leqslant cv_n^2h_n^2\alpha^{\tau/(2+\tau)}\left(\frac{\varrho(\mathbf{k},\mathbf{l})-1}{\Lambda_n}, 1, 1\right). \tag{5.13}$$

By (5.10) and (5.13),

$$\begin{aligned} \pi_n &\leqslant \frac{1}{h_n^2\mu(T_n)} \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathcal{D}_n \\ \varrho(\mathbf{k},\mathbf{l})\geqslant t\Lambda_n+1}} c_{\mathbf{kl}} + \frac{1}{h_n^2\mu(T_n)} \sum_{\substack{\mathbf{k},\mathbf{l}\in\mathcal{D}_n \\ \varrho(\mathbf{k},\mathbf{l})\leqslant t\Lambda_n}} c_{\mathbf{kl}} \\ &\leqslant \frac{cv_n^2|\mathcal{D}_n|}{\mu(T_n)} \sum_{s=t\Lambda_n+1}^{\infty} s^{d-1}\alpha^{\tau/(2+\tau)}\left(\frac{s-1}{\Lambda_n}, 1, 1\right) + \\ &\quad + \frac{cv_n^2|\mathcal{D}_n|}{h_n^2\mu(T_n)}(t\Lambda_n)^d\left(M_n^2\omega_n^\Gamma + V_nM_n\omega_n\right) \\ &\leqslant cv_n\int_{t\Lambda_n}^{\infty} s^{d-1}\alpha^{\tau/(2+\tau)}\left(\frac{s-1}{\Lambda_n}, 1, 1\right)\,\mathrm{d}s + \\ &\quad + cv_nh_n^{-2}(t\Lambda_n)^d\left(M_n^2\omega_n^\Gamma + V_nM_n\omega_n\right) \\ &\leqslant cv_n\Lambda_n^d\int_t^\infty s^{d-1}\alpha^{\tau/(2+\tau)}(s,1,1)\,\mathrm{d}s + cv_nh_n^{-2}(t\Lambda_n)^d\left(M_n^2\omega_n^\Gamma + V_nM_n\omega_n\right) \\ &\leqslant c\int_t^\infty s^{d-1}\alpha^{\tau/(2+\tau)}(s,1,1)\,\mathrm{d}s + ch_n^{-2}t^d\left(M_n^2\omega_n^\Gamma + V_nM_n\omega_n\right). \end{aligned} \tag{5.14}$$

In the above expression first choosing t to be large enough, then taking $n \to \infty$, we obtain (5.9). Therefore, by (2.20) and by Remark 5.1,

$$\liminf_{n\to\infty}\left(h_n^{-2}\Lambda_n^{-d}|\mathcal{D}_n|^{-1}\lambda_{\min}(\Sigma_n)\right) > 0, \tag{5.15}$$

so the central limit theorem in Remark 4.3 implies

$$\left(v_n^2\Sigma_n\right)^{-1/2}(v_n D_n)(\widehat{\beta}_n-\beta)\Rightarrow\mathcal{N}(0,I_p), \tag{5.16}$$

as $n\to\infty$.

To prove (2.23), we have to change $v_n^2\Sigma_n$ in (5.16) for Σ_n^c. However, (5.9) and (2.20) imply that

$$\lim_{n\to\infty}(\lambda_{\min}(\Sigma_n^c))^{-1}\|\Sigma_n^c-v_n^2\Sigma_n\|=0.$$

This together with (5.16) imply that

$$\left(\Sigma_n^c\right)^{-1/2}(v_n D_n)(\widehat{\beta}_n-\beta)\Rightarrow\mathcal{N}(0,I_p) \tag{5.17}$$

(see Lemma 5.3 below for details).

To prove (2.25), we have to substitute $D_n^c=\int_{T_n}f(\mathbf{x})f(\mathbf{x})^\top\,\mathrm{d}\mathbf{x}$ for $(v_n D_n)$ in (5.17). We have

$$\begin{aligned}&\left\|\left(\Sigma_n^c\right)^{-1/2}(v_n D_n)(\widehat{\beta}_n-\beta)-\left(\Sigma_n^c\right)^{-1/2}D_n^c(\widehat{\beta}_n-\beta)\right\|\\&\quad\leqslant\left\|\left(\Sigma_n^c\right)^{-1/2}\right\|\|v_n D_n-D_n^c\|\|\widehat{\beta}_n-\beta\|\\&\quad\leqslant c(h_n^2\mu(T_n))^{-1/2}r_n\|\widehat{\beta}_n-\beta\|.\end{aligned} \tag{5.18}$$

By (2.19), the same calculation as in the proof of Lemma 5.2 we get

$$\mathbb{E}\|\widehat{\beta}_n-\beta\|\leqslant c\mathbb{E}\|K_n\|\leqslant c(\mathbb{E}\|K_n\|^2)^{1/2}\leqslant ch_n g_n^{-1}\mu^{-1/2}(T_n).$$

Therefore the expression in (5.18) is majorized by

$$c\frac{r_n}{g_n\mu(T_n)}.$$

Therefore Slutsky's theorem and (2.24) give (2.25). □

LEMMA 5.3. *Let ξ_n be a sequence of random vectors, let Σ_n be a sequence of symmetric positive definite matrices for which $\Sigma_n^{-1/2}\xi_n\Rightarrow\mathcal{N}(0,I)$, where I is the identity matrix. Let $\breve{\Sigma}_n$ be another sequence of symmetric positive definite matrices for which*

$$\lim_{n\to\infty}(\lambda_{\min}(\breve{\Sigma}_n))^{-1}\|\breve{\Sigma}_n-\Sigma_n\|=0. \tag{5.19}$$

Then $\breve{\Sigma}_n^{-1/2}\xi_n\Rightarrow\mathcal{N}(0,I)$.

Proof. Using characteristic functions it is easy to obtain that $U_n\Sigma_n^{-1/2}\xi_n\Rightarrow\mathcal{N}(0,I)$ for any sequence U_n of orthogonal matrices. Therefore, by the Slutsky theorem, it is sufficient to show that there exists a sequence U_n of orthogonal matrices such that

$$\|\breve{\Sigma}_n^{-1/2}\xi_n-U_n\Sigma_n^{-1/2}\xi_n\|=o\|\Sigma_n^{-1/2}\xi_n\|. \tag{5.20}$$

Now, by (5.19),

$$\|\breve{\Sigma}_n^{-1/2}(\Sigma_n - \breve{\Sigma}_n)\Sigma_n^{-1/2}\| \leqslant c(\lambda_{\min}(\breve{\Sigma}_n))^{-1}\|\breve{\Sigma}_n - \Sigma_n\| \to 0,$$

therefore $\breve{\Sigma}_n^{-1/2}\Sigma_n\breve{\Sigma}_n^{-1/2} \to I$. Using notation $S_n = \breve{\Sigma}_n^{-1/2}\Sigma_n^{1/2}$, we have $S_n S_n^\top \to I$. This relation itself implies that the sequence S_n is bounded, S_n is nonsingular for large n, and $(S_n S_n^\top)^{-1/2} \to I$. Let $U_n = (S_n S_n^\top)^{-1/2} S_n$. Then U_n is orthogonal. Moreover,

$$\breve{\Sigma}_n^{-1/2}\Sigma_n^{1/2} - U_n = S_n - U_n = \left(I - (S_n S_n^\top)^{-1/2}\right) S_n \to 0.$$

This implies $\|\breve{\Sigma}_n^{-1/2}\Sigma_n^{1/2}x - U_n x\| = o(\|x\|)$, and substituting $x = \Sigma_n^{-1/2}\xi_n$ this gives (5.20). □

Remark 5.2. To express the stability of multipliers it is often convenient to write (2.25) into the form

$$\left(\frac{1}{h_n^2\mu(T_n)}\Sigma_n^c\right)^{-1/2}\left(\frac{1}{g_n\mu(T_n)}D_n^c\right)\left(\frac{g_n}{h_n}\sqrt{\mu(T_n)}(\widehat{\beta}_n - \beta)\right) \Rightarrow \mathcal{N}(0, I_p). \tag{5.21}$$

6. Proofs for the Errors-in-Variables Model

Proof of Theorem 3.1. According to Lemma 5.1, if n is large enough, $\tilde{D}_n = \sum_{\mathbf{k}\in\mathcal{D}_n} f(\mathbf{u_k})f(\mathbf{u_k})^\top$ is invertible. Then, by (3.6) and (3.1),

$$\tilde{\beta}_n = \tilde{D}_n^{-1}\sum_{\mathbf{k}\in\mathcal{D}_n} z(\mathbf{x_k})f(\mathbf{u_k}) = C_n\beta + \left(\frac{v_n}{g_n\mu(T_n)}\tilde{D}_n\right)^{-1}\tilde{K}_n, \tag{6.1}$$

where

$$\tilde{K}_n = \frac{v_n}{g_n\mu(T_n)}\sum_{\mathbf{k}\in\mathcal{D}_n}\varepsilon(\mathbf{x_k})f(\mathbf{u_k}), \qquad C_n = \tilde{D}_n^{-1}\sum_{\mathbf{k}\in\mathcal{D}_n} f(\mathbf{u_k})f(\mathbf{x_k})^\top. \tag{6.2}$$

In Lemma 5.2 we used only the bound of the function f, and not the precise value of $f(\mathbf{x_k})$, therefore we conclude that $\tilde{K}_n \to 0$ in probability. Therefore, using also Lemma 5.1, the second term at the right hand side of (6.1) converges to 0 in probability, as $n \to \infty$. So it is enough to prove that C_n converges to the identity matrix $I = I_p$. Let $J_n = (1/g_n\mu(T_n))\int_{T_n} f(\mathbf{x})f(\mathbf{x})^\top\,\mathrm{d}\mathbf{x}$. By (2.6), J_n is invertible. We have

$$\begin{aligned} C_n &= \left(\frac{v_n}{g_n\mu(T_n)}\tilde{D}_n\right)^{-1}\left(\frac{v_n}{g_n\mu(T_n)}\sum_{\mathbf{k}\in\mathcal{D}_n} f(\mathbf{u_k})f(\mathbf{x_k})^\top\right) \\ &= (J_n + U_n)^{-1}(J_n + W_n) = (I + J_n^{-1}U_n)^{-1}(I + J_n^{-1}W_n). \end{aligned}$$

Therefore

$$C_n - I = (I + J_n^{-1}U_n)^{-1}J_n^{-1}(W_n - U_n). \tag{6.3}$$

By (2.6), $\|J_n^{-1}\| \leqslant 1/\lambda_0 < \infty$. Moreover,

$$\begin{aligned} \|W_n - U_n\| &\leqslant \frac{v_n}{g_n\mu(T_n)} \sum_{\mathbf{k}\in\mathcal{D}_n} \|f(\mathbf{u_k}) - f(\mathbf{x_k})\|\cdot\|f(\mathbf{u_k})\| \\ &\leqslant \frac{\mu(T_n^\circ \cup \delta T_n)}{g_n\mu(T_n)} M_n\omega_n \to 0, \end{aligned}$$

as $n \to \infty$, because of (2.8). By (5.4), $\lim_{n\to\infty}\|U_n\| = 0$. Therefore, if n is large enough, $\|J_n^{-1}U_n\| < c < 1$. For such n we get $\|(I + J_n^{-1}U_n)^{-1}\| \leqslant (1 - \|J_n^{-1}U_n\|)^{-1}$. Therefore (6.3) implies that C_n converges to identity matrix. □

Proof of Theorem 3.2. According to (3.5)

$$\tilde{D}_n(\tilde{\beta}_n - \beta) = \sum_{\mathbf{k}\in\mathcal{D}_n}[f(\mathbf{u_k})f^\top(\mathbf{x_k}) - f(\mathbf{u_k})f^\top(\mathbf{u_k})]\beta + \sum_{\mathbf{k}\in\mathcal{D}_n}\varepsilon(\mathbf{x_k})f(\mathbf{u_k}). \tag{6.4}$$

Let

$$\tilde{\Sigma}_n = \operatorname{var}\left(\sum_{\mathbf{k}\in\mathcal{D}_n}\varepsilon(\mathbf{x_k})f(\mathbf{u_k})\right) = \sum_{\mathbf{k},\mathbf{l}\in\mathcal{D}_n}\Gamma(\mathbf{x_k},\mathbf{x_l})f(\mathbf{u_k})f(\mathbf{u_l})^\top. \tag{6.5}$$

First we prove that

$$\left(h_n^{-2}\tilde{\Sigma}_n\right)^{-1/2} h_n^{-1}\sum_{\mathbf{k}\in\mathcal{D}_n}\varepsilon(\mathbf{x_k})f(\mathbf{u_k}) \Rightarrow \mathcal{N}(0, I_p). \tag{6.6}$$

We have to prove (4.17). By condition (2.5), to this end it is sufficient to prove that

$$\liminf_{n\to\infty}\lambda_{\min}\left(h_n^{-2}(\mu(T_n))^{-1}v_n^2\tilde{\Sigma}_n\right) > 0. \tag{6.7}$$

We shall prove that

$$\tilde{\pi}_n = (h_n^2\mu(T_n))^{-1}\left\|v_n^2\tilde{\Sigma}_n - \Sigma_n^c\right\| \to 0, \tag{6.8}$$

as $n \to \infty$. By (3.9), $h_n^{-2}|\Gamma(\mathbf{x},\mathbf{y})|M_n(\mathbf{x})M_n(\mathbf{y})$ is bounded, therefore (by (2.5)) it is sufficient to consider the case when T_n is a union of the hyperrectangles $\Delta_n(\mathbf{k})$. Then

$$\tilde{\pi}_n \leqslant \frac{1}{h_n^2\mu(T_n)}\sum_{\mathbf{k},\mathbf{l}\in\mathcal{D}_n}\tilde{c}_{\mathbf{kl}},$$

where

$$\tilde{c}_{\mathbf{kl}} = \left\| \int_{\Delta_n(\mathbf{k})} \int_{\Delta_n(\mathbf{l})} \Gamma(\mathbf{x}, \mathbf{y}) f(\mathbf{x}) f(\mathbf{y})^\top \,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{y} - v_n^2 \Gamma(\mathbf{x_k}, \mathbf{x_l}) f(\mathbf{u_k}) f(\mathbf{u_l})^\top \right\|.$$

Like in inequality (5.10) we have

$$\tilde{c}_{\mathbf{kl}} \leqslant c v_n^2 (M_n^2 \omega_n^\Gamma + V_n M_n \omega_n). \tag{6.9}$$

Using the covariance inequality and condition (3.9) we get

$$v_n^2 \|\Gamma(\mathbf{x_k}, \mathbf{x_l}) f(\mathbf{u_k}) f(\mathbf{u_l})^\top\| \leqslant c v_n^2 h_n^2 \alpha^{\tau/(2+\tau)}(\varrho_{\mathbf{k},\mathbf{l}}, 1, 1), \tag{6.10}$$

where $\varrho_{\mathbf{k},\mathbf{l}} = \varrho\,(\Delta_n(\mathbf{k}), \Delta_n(\mathbf{l}))$. This and (5.11) imply that

$$\tilde{c}_{\mathbf{kl}} \leqslant c v_n^2 h_n^2 \alpha^{\tau/(2+\tau)}(\varrho_{\mathbf{k},\mathbf{l}}, 1, 1) \leqslant c v_n^2 h_n^2 \alpha^{\tau/(2+\tau)} \left(\frac{\varrho(\mathbf{k}, \mathbf{l}) - 1}{\Lambda_n}, 1, 1 \right). \tag{6.11}$$

Like in (5.14), by (6.9) and (6.11),

$$\tilde{\pi}_n \leqslant c \int_t^\infty s^{d-1} \alpha^{\tau/(2+\tau)}(s, 1, 1)\,\mathrm{d}s + c h_n^{-2} t^d \left(M_n^2 \omega_n^\Gamma + V_n M_n \omega_n \right). \tag{6.12}$$

Therefore we obtain (6.8) and then (6.7). So the central limit theorem in Remark 4.3 implies (6.6). In (6.6) we have to change $v_n^2 \tilde{\Sigma}_n$ for Σ_n^c. However, (6.8) and Lemma 5.3 imply the desired relation:

$$\left(\Sigma_n^c\right)^{-1/2} v_n \sum_{\mathbf{k} \in \mathcal{D}_n} \varepsilon(\mathbf{x_k}) f(\mathbf{u_k}) \Rightarrow \mathcal{N}(0, I_p). \tag{6.13}$$

Now, to obtain (3.11), we have to prove

$$\lim_{n \to \infty} \left(\Sigma_n^c\right)^{-1/2} v_n \sum_{\mathbf{k} \in \mathcal{D}_n} \left(f(\mathbf{u_k}) f^\top(\mathbf{x_k}) - f(\mathbf{u_k}) f^\top(\mathbf{u_k}) \right) \beta = 0. \tag{6.14}$$

However, (3.10) implies it immediately.

To prove (3.14) we have to substitute $D_n^c = \int_{T_n} f(\mathbf{x}) f(\mathbf{x})^\top \,\mathrm{d}\mathbf{x}$ for $(v_n D_n)$ in (3.11). By (3.12) and (3.13)

$$\lim_{n \to \infty} \left\| \left(\Sigma_n^c\right)^{-1/2} (v_n \tilde{D}_n)(\tilde{\beta}_n - \beta) - \left(\Sigma_n^c\right)^{-1/2} D_n^c (\tilde{\beta}_n - \beta) \right\| = 0 \tag{6.15}$$

in probability. (It is proved by the same calculation as in the proof of Theorem 2.2.) Therefore Slutsky's theorem gives (3.14). □

7. Example

EXAMPLE 7.1. Consider the model

$$z(\mathbf{x}) = \beta^\top \mathbf{x} + \varepsilon(\mathbf{x}), \quad \mathbf{x} \in T_\infty. \tag{7.1}$$

Suppose that we observe the random field $z(\mathbf{x})$ at points $\mathbf{x_k} \in T_n$, where $\mathbf{k} \in \mathcal{D}_n$. Let $T_n = [-n, n]^d$, $n = 1, 2, \ldots$. Therefore, $T_\infty = \mathbb{R}^d$. We have $\mu(T_n) = (2n)^d$,

$$D_n^c = \int_{T_n} \mathbf{x}\mathbf{x}^\top \, \mathrm{d}\mathbf{x} = cn^{d+2} I_d. \tag{7.2}$$

Now we shall describe conditions sufficient to Theorem 2.1. We divide T_n into hyperrectangles $\Delta_n(\mathbf{k})$ such that $\delta T_n = \emptyset$, therefore (2.5) and (2.7) are satisfied. Choose $g_n = n^2$. Then $g_n^{-1}\mu^{-1}(T_n)D_n^c = cI_d$, therefore (2.6) is valid. We have $M_n\omega_n g_n^{-1} = c\omega_n n^{-1}$, therefore if we suppose that $\lim_{n\to\infty} \omega_n n^{-1} = 0$ (which means that the division into hyperrectangles is 'dense enough') then (2.8) is satisfied. We can suppose that (2.21) is also satisfied (we divide T_n into cubes, say). In this case

$$\frac{v_n^{1-\tau} M_n^{2-\tau} V_n^{(2-\tau)/2} \Lambda_n^d}{g_n^{2-\tau}\mu^{1-\tau}(T_n)} \leqslant c \frac{v_n^{-\tau} V_n^{(2-\tau)/2}}{n^{2-\tau} n^{d(1-\tau)}}. \tag{7.3}$$

This implies that the boundedness of V_n and $\lim_{n\to\infty} v_n^\tau n^{2-\tau+d(1-\tau)} = \infty$ (which means that the measure of the whole domain tends to infinity fast enough compared to the density of its division into hyperrectangles) are sufficient for (2.11). A sufficient condition for (2.9) is

$$\alpha(s, 1, 1) \leqslant cs^{(-d(2+\tau)/\tau)-t} \quad \text{for some} \quad t > 0. \tag{7.4}$$

Now we turn to Theorem 2.2. Concerning mixing conditions we suppose that (7.4) is satisfied and

$$\alpha(s, i, j) \leqslant cs^{-d-t} \quad \text{for} \; i + j \leqslant 4 \; \text{for some} \; t > 0, \tag{7.5}$$

$$\alpha(s, 1, \infty) \leqslant cs^{-d-t} \quad \text{for some} \; t > 0. \tag{7.6}$$

These conditions are sufficient for (2.9), (2.17) and (2.18). Suppose that $h_n = n$ and $\mathbb{E}|\varepsilon(\mathbf{x})|^{2+\tau+t}$ is bounded for some $t > 0$. Then

$$\{\left(h_n^{-1}\|f(\mathbf{x})\|\cdot|\varepsilon(\mathbf{x})|\right)^{2+\tau} : \mathbf{x} \in T_n, \; n = 1, 2, \ldots\}$$

are uniformly integrable, that is (2.19) is satisfied. Now

$$h_n^{-2}(M_n^2\omega_n^\Gamma + V_n M_n \omega_n) = \omega_n^\Gamma + V_n\omega_n/n, \tag{7.7}$$

therefore if $\varepsilon(\mathbf{x})$ is L^2-continuous, V_n is bounded, and $\lim_{n\to\infty} \omega_n n^{-1} = 0$, then (2.22) is satisfied.

Now, we specify the covariance function as

$$\Gamma(\mathbf{x}, \mathbf{y}) = c \exp\{-\|\mathbf{x} - \mathbf{y}\|^2/2\}. \tag{7.8}$$

Then

$$\lim_{n\to\infty} \frac{1}{h_n^2 \mu(T_n)} \Sigma_n^c = cI_p. \tag{7.9}$$

Therefore, (2.20) is satisfied. (We remark that the covariance function (7.8) is continuous, and in this case the boundedness of V_n and (7.4) are consequences of (7.8).) By Theorem 2.2, the above assumptions imply asymptotic normality of $\widehat{\beta}_n$, i.e. (2.23). Concerning condition (2.24) we remark that

$$\frac{r_n}{g_n \mu(T_n)} \leqslant c \frac{v_n}{n}. \tag{7.10}$$

The right hand side converges to 0, therefore (2.24) is satisfied, so Theorem 2.2 implies the asymptotic normality relation (2.25), too.

8. Simulation Results

In Examples 8.1, 8.2, and 8.3 we study the behaviour of the least square estimator in three different cases. I. Dense observations in a large domain. II. Distant observations in a large domain. III. Dense observations in a small domain. It will turn out that the first two cases are similar but in the third case the variance of the estimator is much larger than in the first and second ones.

The simulations were performed with MATLAB. In each cases 1000 repetitions were made. The data sets in cases II and III were subsets of the data set in case I.

EXAMPLE 8.1. Let the random field $\varepsilon(\mathbf{x})$ be a spatial moving average process on the planar grid with step 0.05:

$$\varepsilon(x_1, x_2) = \sum_{i=-20}^{20} \sum_{j=-20}^{20} s(il_1, jl_2) W(x_1 - il, x_2 - jl), \tag{8.1}$$

where x_1 and x_2 are of the form $x_1 = il$, $x_2 = jl$, l is fixed, i and j are integers. Here $W(x_1, x_2)$ is a discrete parameter Gaussian white noise on the planar grid with $\mathbb{E}W(x_1, x_2) = 0$ and $\mathbb{D}^2 W(x_1, x_2) = 0.04^2$. The weight function is $s(u, v) = e^{-(u^2+v^2)/2}$. l, l_1, and l_2 are chosen as 0.05.

The observed field is

$$z(x_1, x_2) = \beta_1 x_1 + \beta_2 x_2 + \varepsilon(x_1, x_2). \tag{8.2}$$

Therefore, z can be considered as a discrete approximation of the field in Example 7.1. We chose $\beta_1 = 1$ and $\beta_2 = -1$. We observed the process z on the grid with step h on the square $[-th, th] \times [-th, th]$.

Three sets of observations were taken.

I. Dense observations in a large square: $t = 40$, $h = 0.1$.
II. Distant observations in a large square: $t = 10$, $h = 0.4$.
III. Dense observations in a small square: $t = 10$, $h = 0.1$.

Table I and Figure 1 show that the estimators are unbiased and asymptotic normal. However, the variance of the estimator in the third case is much larger

Table I. Results in Example 8.1

Case	Number of observations	Distance of observations	Mean($\widehat{\beta}_1$)	Var($\widehat{\beta}_1$)	Mean($\widehat{\beta}_2$)	Var($\widehat{\beta}_2$)
I	6561	0.1	0.9950	0.0119	−1.0013	0.0112
II	441	0.4	0.9951	0.0107	−1.0013	0.0100
III	441	0.1	0.9944	0.6147	−1.0099	0.6148

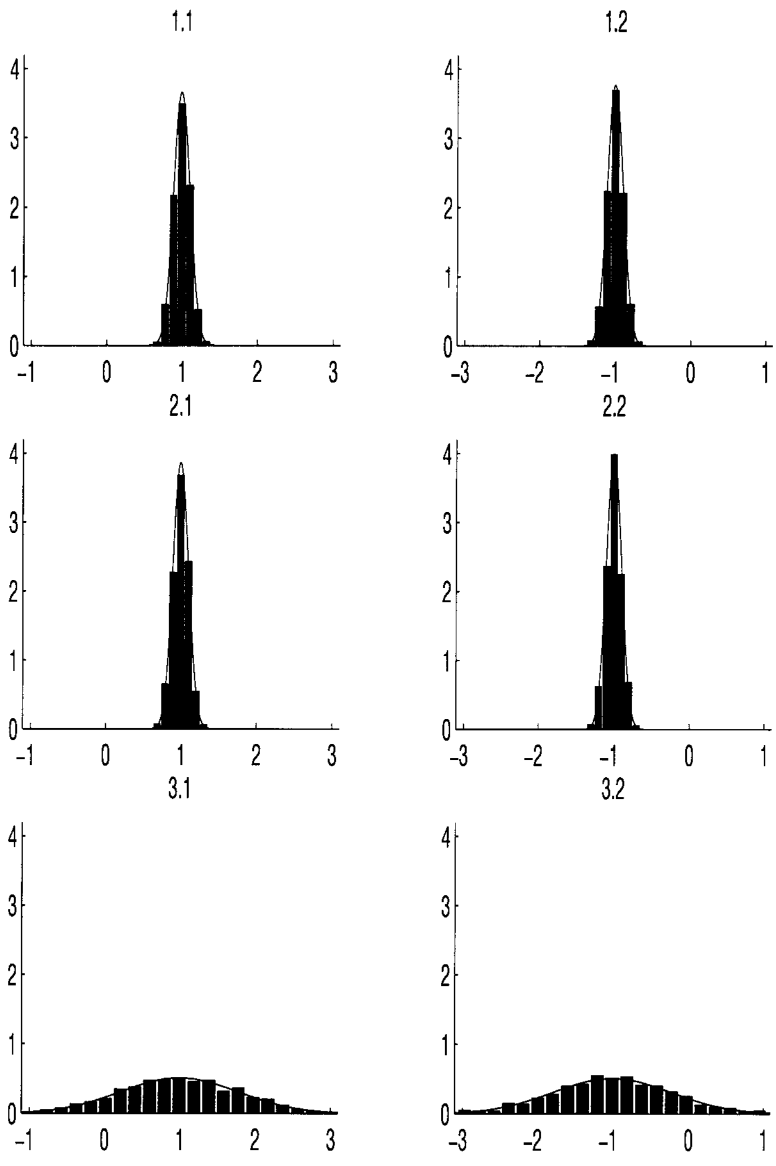

Figure 1. Results in Example 8.1.

Table II. Results in Example 8.2

Case	Number of observations	Distance of observations	Mean($\widehat{\beta}_1$)	Var($\widehat{\beta}_1$)	Mean($\widehat{\beta}_2$)	Var($\widehat{\beta}_2$)
I	6561	0.1	0.9977	0.0024	−1.0009	0.0020
II	441	0.4	0.9980	0.0022	−1.0007	0.0018
III	441	0.1	0.9934	0.2632	−1.0069	0.1149

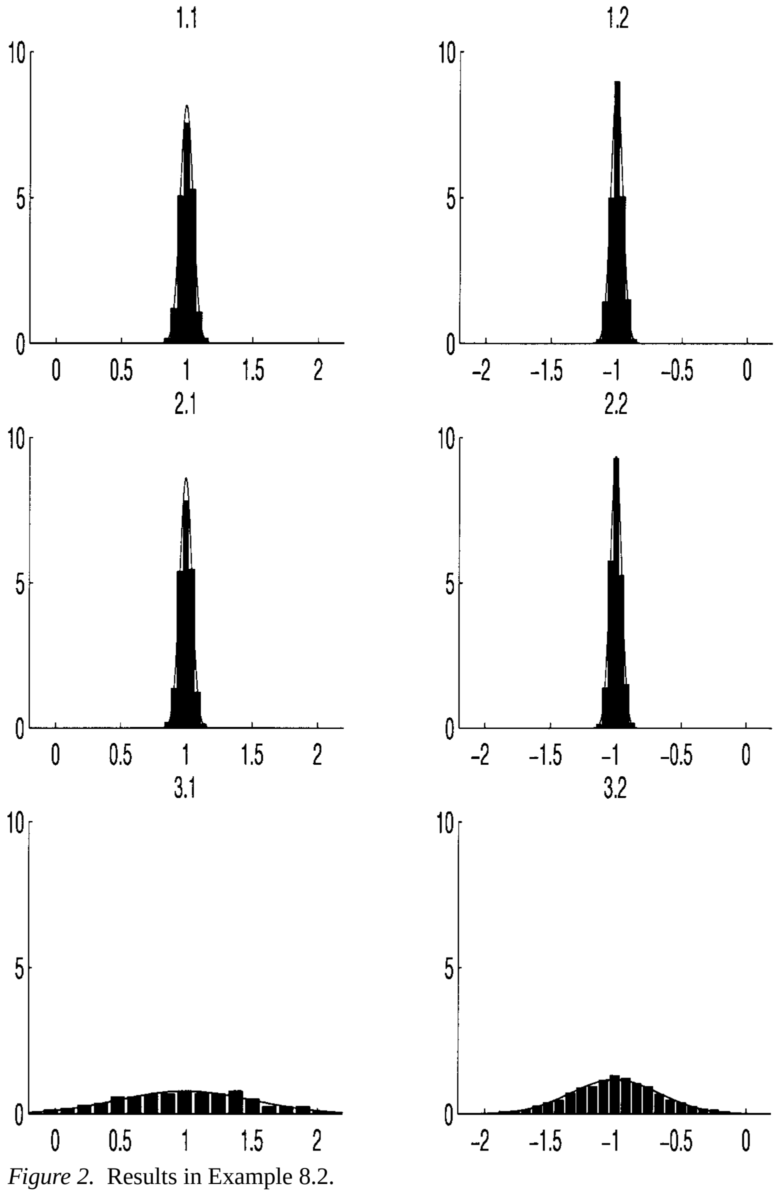

Figure 2. Results in Example 8.2.

Table III. Results in Example 8.3

Case	Number of observations	Distance of observations	Mean($\widehat{\beta}_1$)	Var($\widehat{\beta}_1$)	Mean($\widehat{\beta}_2$)	Var($\widehat{\beta}_2$)
I	6561	0.1	1.0000	1.062×10^{-6}	-1.0000	1.118×10^{-6}
II	441	0.4	1.0000	3.878×10^{-6}	-0.9999	3.743×10^{-6}
III	441	0.1	0.9988	173.6×10^{-6}	-0.9996	221.3×10^{-6}

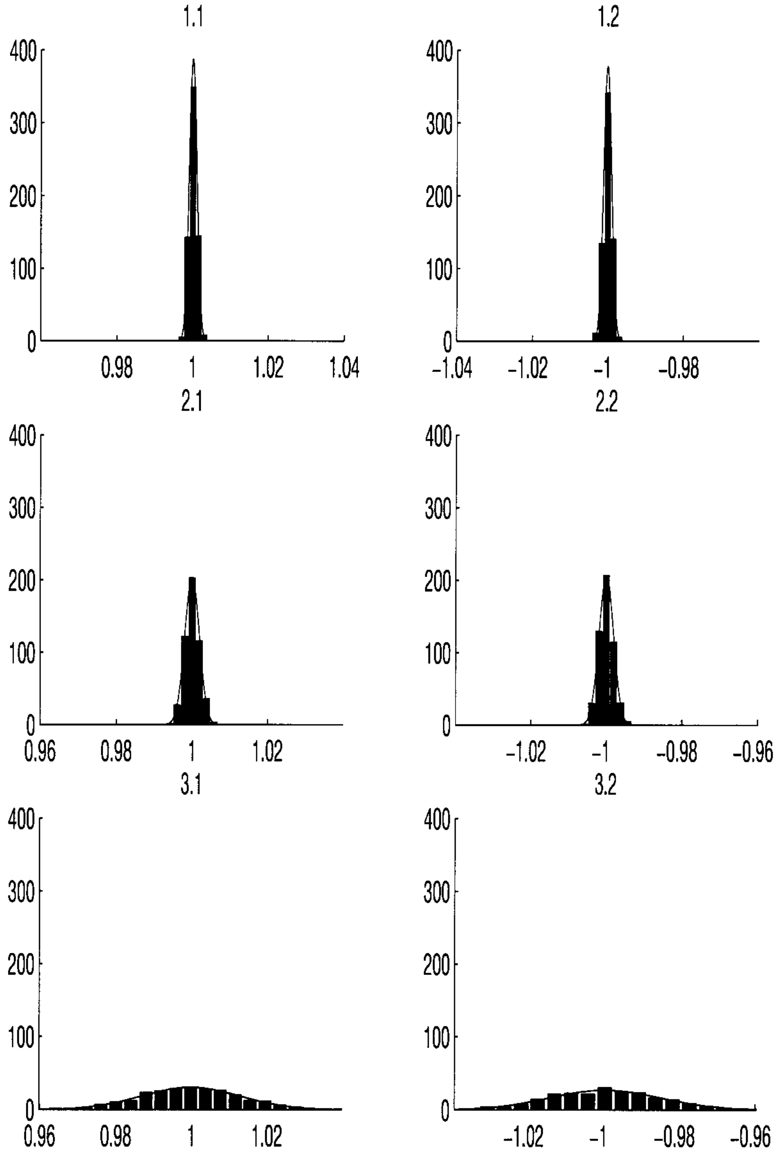

Figure 3. Results in Example 8.3.

than in the first and second cases. Remark that on Figure 1 subplots 1.1 and 1.2 correspond to $\widehat{\beta}_1$ and $\widehat{\beta}_2$, respectively, in case I. Similarly, subplots 2.1 and 2.2 correspond to case II, while subplots 3.1 and 3.2 correspond to case III.

EXAMPLE 8.2. The observed random field is similar to the one in Example 8.1. The only difference is that $l_1 = 0.01$ and $l_2 = 0.2$, that is the dependence is stronger in the horizontal direction than in the vertical direction. The scheme of observations and the parameters are the same as in Example 8.1. The results are shown in Table II and Figure 2.

EXAMPLE 8.3. It is a modification of Example 8.1. The dependence is much weaker (especially in vertical direction) than in Example 8.1. Let

$$\varepsilon(x_1, x_2) = \sum_{i=-60}^{60} \sum_{j=-10}^{10} s(il_1, jl_2) W(x_1 - il, x_2 - jl), \tag{8.3}$$

where x_1 and x_2 are of the form $x_1 = il$, $x_2 = jl$, $l = 0.1$ is fixed, i and j are integers, $l_1 = 0.5$, $l_2 = 4$. The white noise W, the weight function s, the definition of z by Equation (8.2), and the observation cases I–II–III are the same as in Example 8.1.

Table III and Figure 3 show that the estimators are unbiased and asymptotic normal. The variance of the estimator in the third case is much larger than in the first and second cases. Because of the weak dependence, the dispersion in the first case is less than in the second case.

Further examples are presented in Fazekas and Kukush [4].

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