

See discussions, stats, and author profiles for this publication at: <https://www.researchgate.net/publication/323640285>

Comonotonic asset prices in arbitrage-free markets

Preprint · March 2018

DOI: 10.13140/RG.2.2.33575.24485

CITATIONS

0

3 authors, including:



Jan Dhaene

University of Leuven

273 PUBLICATIONS 5,410 CITATIONS

SEE PROFILE



Daniël Linders

University of Illinois, Urbana-Champaign

41 PUBLICATIONS 200 CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:



The valuation of risk aggregation with applications in finance and insurance [View project](#)



Risk Measures and Stochastic Ordering [View project](#)

All content following this page was uploaded by [Daniël Linders](#) on 08 March 2018.

The user has requested enhancement of the downloaded file.

Comonotonic asset prices in arbitrage-free markets

Jan Dhaene* Alexander Kukush† Daniël Linders‡

February 20, 2018

Abstract

For an arbitrage-free market with a single underlying asset, we investigate conditions under which the consecutive price levels are comonotonic. Furthermore, for an arbitrage-free market with n assets we investigate the consequences of assuming comonotonicity of the vector containing the price levels of each asset at a single future date T . Although being of a theoretical nature, the results of this paper give insight in the reachability of the comonotonic upper bounds for Asian options and for basket options that can be found in Simon et al. (2000), Dhaene et al. (2002b), Hobson et al. (2005) and Chen et al. (2008).

1 Introduction

Random variables (r.v.'s) are said to be comonotonic if they are all non-decreasing (or non-increasing) functions of the same random source. As a result, comonotonicity describes a situation where the r.v.'s are moving perfectly together. The concept of comonotonicity has been investigated in detail in the actuarial and financial literature; see e.g. Dhaene et al. (2002a,b), Deelstra et al. (2011), Cheung (2007, 2009, 2010). Assuming a comonotonic dependence structure between asset prices has proven to be a useful tool to solve several financial applications such as derivative pricing (Deelstra et al. (2004), Deelstra et al. (2008) and Linders & Stassen (2016)) and dependence measurement; see e.g. Dhaene et al. (2012a), Dhaene et al. (2012b), Dhaene et al. (2014a) and Linders et al. (2012), to name a few.

In this paper, we first consider an arbitrage-free market consisting of a risk-free bank account and a single risky asset. Today is time $t = 0$ and we consider the random price levels $S(t_1), S(t_2), \dots, S(t_n)$ at the discrete set of times $0 < t_1 < t_2 < \dots < t_n$. We derive conditions under which the real world as well as the risk-neutral distribution of

*KU Leuven, Leuven, Belgium. Email: Jan.Dhaene@kuleuven.be

†Faculty of Mechanics and Mathematics, Taras Shevchenko National University of Kyiv, Kyiv, Ukraine. Email: alexander_kukush@univ.kiev.ua

‡University of Illinois at Urbana-Champaign, Urbana, USA. Email: dlinders@illinois.edu

the random vector $(S(t_1), S(t_2), \dots, S(t_n))$ can be comonotonic. To be more precise, we prove that for popular continuous models such as the Black & Scholes model, the comonotonicity property is in contradiction with the no-arbitrage assumption. We also provide examples of stock price models where the price vector $(S(t_1), S(t_2), \dots, S(t_n))$ may be comonotonic, without contradicting the no-arbitrage assumption.

Consider the *Asian option* with strike K and maturity T whose pay-off at maturity is equal to $(\frac{1}{n} \sum_{i=1}^n S(t_i) - K)_+$. Knowledge about the dynamics of the price vector $(S(t_1), S(t_2), \dots, S(t_n))$ is required in order to price this Asian option. Suppose that also vanilla options on the stock S with maturity t_i , $i = 1, 2, \dots, n$ are traded. Their prices give information about the risk-neutral cdfs of the $S(t_i)$. However, this information is not sufficient to specify the risk-neutral cdf of $(S(t_1), S(t_2), \dots, S(t_n))$. One may search for an upper bound of the price of an Asian option in the class of models which are consistent with the traded vanilla option prices; see e.g. Kerkhof et al. (2010) and Barrieu & Scandolo (2015). In case comonotonicity of $(S(t_1), S(t_2), \dots, S(t_n))$ does not violate the no arbitrage assumption, this ‘worst case price’ can be derived using the prices of traded vanilla options and then coincides with the comonotonic upper bound for the Asian option price which was derived in Simon et al. (2000), Dhaene et al. (2002b) and Albrecher et al. (2005). We show that for continuous stock price models such as the Black & Scholes model and a variety of Lévy models, a comonotonic stock price vector is not attainable. The comonotonic upper bound for the Asian option will be free of any model risk, but not reachable.

Next, we consider an arbitrage-free market consisting of n traded (dividend or non-dividend paying) assets and a risk-free bank account. The price of asset i at time T is denoted by S_i . We explore situations under which the price vector (S_1, S_2, \dots, S_n) is comonotonic and arbitrage-free. In particular, we have a closer look at the multivariate Black & Scholes model and give examples of comonotonic price vectors. In Simon et al. (2000), Dhaene et al. (2002b) and Chen et al. (2008), the authors show that the situation where the price vector (S_1, S_2, \dots, S_n) is comonotonic can be artificially created using vanilla option data.

A *basket option* written on the stocks S_1, S_2, \dots, S_n with strike K , maturity T has a pay-off at maturity equal to $(\sum_{i=1}^n w_i S_i - K)_+$, where all $w_i \geq 0$. Pricing the basket option requires knowledge about the marginal cdf’s and the dependence between the components. Similar to the Asian option case, any misspecification of the dependence structure leads to a wrongly priced basket option and one can try to reduce the model-risk by determining the basket option price in a ‘worst case scenario’ among the class of models which are consistent with the observed vanilla option prices. In this paper, we show that in case of a basket option, this worst case scenario is always described by comonotonicity of the stock price vector (S_1, S_2, \dots, S_n) . Furthermore, the price of the basket option in this comonotonic scenario is always free of arbitrage and is equal to an upper bound which is free of any model risk.

This paper is structured as follows. In Section 2 we briefly recapitulate the concept of comonotonicity and some of its properties that we will need for proving our main results. In Section 3 we describe the general financial market in which our results hold. In Section 4 we first present results concerning the comonotonicity of the asset prices of a single asset

at different dates. We consider the concept of comonotonicity of the prices of different assets at a single date in Section 5. Section 6 concludes the paper.

2 Comonotonic random variables

In the sequel, the vector (x_1, \dots, x_n) will be denoted by \underline{x} . For two vectors \underline{x} and \underline{y} the notation $\underline{x} \leq \underline{y}$ will be used for the componentwise order, which is defined by $x_i \leq y_i$ for all $i = 1, \dots, n$. Equalities such as $X = Y$ for the r.v.'s X and Y defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ have to be interpreted as almost sure relationships. Finally, $X \stackrel{d}{=} Y$ means that X and Y are equal in distribution.

Definition 1 *A set $A \subseteq \mathbb{R}^n$ is said to be comonotonic if for any \underline{x} and \underline{y} in A , either $\underline{x} \leq \underline{y}$ or $\underline{y} \leq \underline{x}$ holds.*

Hereafter we restate the definition of the notion of comonotonicity of a random vector, as presented in Kaas et al. (2000).

Definition 2 *A random vector \underline{X} is comonotonic if it has a comonotonic support.*

The standard notation F_X^{-1} will be used to denote the usual inverse of the cumulative distribution function (cdf) F_X of the r.v. X

$$F_X^{-1}(p) = \inf \{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in [0, 1], \quad (1)$$

with $\inf \emptyset = +\infty$ by convention.

In the following theorem, several characterizations of the comonotonicity of a random vector are presented. For a proof, we refer to Denneberg (1997) or Dhaene et al. (2002a).

Theorem 3 *A random vector $\underline{X} = (X_1, X_2, \dots, X_n)$ is comonotonic if, and only if, one of the following equivalent conditions holds:*

(1) *For all $\underline{x} = (x_1, x_2, \dots, x_n)$, we have that*

$$F_{\underline{X}}(\underline{x}) = \min \{F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_n}(x_n)\}. \quad (2)$$

(2) *For $U \sim \text{Uniform}(0, 1)$, we have that*

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \dots, F_{X_n}^{-1}(U)). \quad (3)$$

(3) *There exists a r.v. Z and non-decreasing functions $f_i : \mathbb{R} \rightarrow \mathbb{R}, i = 1, \dots, n$, such that*

$$\underline{X} \stackrel{d}{=} (f_1(Z), f_2(Z), \dots, f_n(Z)). \quad (4)$$

(4) There exist continuous and non-decreasing functions $f_i : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f_1(z) + f_2(z) + \dots + f_n(z) = z \quad (5)$$

for any $z \in \mathbb{R}$ and

$$\underline{X} = (f_1(S), f_2(S), \dots, f_n(S)), \quad (6)$$

with

$$S = X_1 + X_2 + \dots + X_n. \quad (7)$$

In Expression (6) and below we assume that the equality between the r.v.'s holds $\mathbb{P} - a.s.$ We present some equivalent characterizations for a comonotonic random vector of which at least one component has a continuous cdf. These characterizations will be used to prove our main results. Without loss of generality, we will assume that the first component of the random vector has a continuous cdf.

Theorem 4 Consider a random vector $\underline{X} = (X_1, X_2, \dots, X_n)$ and assume that X_1 has a continuous cdf. In this case, the following statements are equivalent:

(1) \underline{X} is comonotonic.

(2) There exist non-decreasing functions $f_i, i = 2, 3, \dots, n$, such that

$$\underline{X} \stackrel{d}{=} (X_1, f_2(X_1), f_3(X_1), \dots, f_n(X_1)).$$

(3) There exist non-decreasing functions $f_i, i = 2, 3, \dots, n$, such that

$$\underline{X} = (X_1, f_2(X_1), f_3(X_1), \dots, f_n(X_1)).$$

Proof. (1) \Rightarrow (2) : As F_{X_1} is continuous, we have that $F_{X_1}(X_1)$ is uniformly distributed on the unit interval. Consequently, we also have that

$$F_{X_i}^{-1}(F_{X_1}(X_1)) \stackrel{d}{=} X_i, \quad i = 2, 3, \dots, n.$$

The comonotonic random vector $(X_1, F_{X_2}^{-1}(F_{X_1}(X_1)), F_{X_3}^{-1}(F_{X_1}(X_1)), \dots, F_{X_n}^{-1}(F_{X_1}(X_1)))$ has the same marginals as the comonotonic vector \underline{X} . This implies that

$$\underline{X} \stackrel{d}{=} (X_1, F_{X_2}^{-1}(F_{X_1}(X_1)), F_{X_3}^{-1}(F_{X_1}(X_1)), \dots, F_{X_n}^{-1}(F_{X_1}(X_1))).$$

We find that $\underline{X} \stackrel{d}{=} (X_1, f_2(X_1), \dots, f_n(X_1))$ with $f_i(x) = F_{X_i}^{-1}(F_{X_1}(x)), x \in \mathbb{R}$.

(2) \Rightarrow (1) : Trivial.

(2) \Leftrightarrow (3) : The \Leftarrow implication is straightforward. It remains to prove that (2) implies (3).

Therefore, assume that $\underline{X} \stackrel{d}{=} (X_1, f_2(X_1), \dots, f_n(X_1))$. Then for any $i = 2, 3, \dots, n$, we find that $(X_1, X_i) \stackrel{d}{=} (X_1, f_i(X_1))$. Hence for any $i = 2, 3, \dots, n$, it holds that $(f_i(X_1), X_i) \stackrel{d}{=} (f_i(X_1), f_i(X_1))$, which in turn implies that $X_i = f_i(X_1)$.

We can conclude that $\underline{X} = (X_1, f_2(X_1), \dots, f_n(X_1))$. ■

Theorem 4 is closely related to the concept of *strict comonotonicity*, defined in Chen et al. (2015). The random vector \underline{X} is strictly comonotonic if it has a support such that for any elements $\underline{x}, \underline{y}$ of this support with $\underline{x} \neq \underline{y}$, it must hold that $\underline{x} < \underline{y}$ or $\underline{x} > \underline{y}$. One can prove that in case all marginals have continuous cdf's, comonotonicity and strict comonotonicity are equivalent. From Theorem 4 we find that in this situation, comonotonicity of a random vector implies that the outcome of any random variable determines all other outcomes.

Theorem 4 will not necessary hold in case X_1 has a discrete cdf. Indeed, consider the random couple (X_1, X_2) and suppose that

$$X_{j+1} = X_j Z_{j+1}, \quad j = 0, 1,$$

where X_0 is a given real number and Z_1 and Z_2 are i.i.d. r.v.'s which can only take two different values $u > 0$ and $d > 0$, with $d < u$ and

$$\mathbb{P}[Z_j = u] = 1 - \mathbb{P}[Z_j = d] = p,$$

for some $0 < p < 1$. The couple (X_1, X_2) is comonotonic since it has the following comonotonic support:

$$\{(X_0d, X_0d^2), (X_0d, X_0du), (X_0u, X_0ud), (X_0u, X_0u^2)\}.$$

As X_1 has a two-point distribution, while X_2 has a three-point distribution, it is impossible to find a non-decreasing functions f such that $X_2 = f(X_1)$. This means that the outcome of X_1 does not determine unambiguously the realization of X_2 .

3 The financial market

Consider the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ to describe a financial market consisting of n traded stocks and a bank account. We assume that the filtered probability space satisfies the usual 'technical' conditions of completeness and right-continuity. Moreover, \mathcal{F}_0 contains all \mathbb{P} -null sets of Ω and $\mathcal{F}_T = \mathcal{F}$. Money can be deposited on or borrowed from the bank at the continuously compounded interest intensity $r \geq 0$, which is assumed to be deterministic and constant over time.

Consider a particular stock and suppose that its current price is denoted by $S(0) > 0$, which is assumed to be known. The price of this stock at future dates $t > 0$ is described by the adapted stochastic process $\{S(t) \mid 0 \leq t \leq T\}$ which is defined on the above mentioned filtered probability space. The stock pays a continuous dividend yield ν , which is assumed to be deterministic and constant over time. Denoting the stochastic value of the (non-discounted) dividends paid in $(0, t)$ by $D(t)$, we have that

$$dD(t) = \nu S(t) dt. \tag{8}$$

The gains process $\{G(t) \mid 0 \leq t \leq T\}$ is defined by

$$G(t) = S(t) + \int_0^t e^{r(t-u)} dD(u), \quad t > 0, \tag{9}$$

with initial value $G(0) = S(0)$. If $\{S(t) \mid t \geq 0\}$ has continuous paths, then the integral in (9) is a path-wise Riemann integral, a.s.

Loosely speaking, the financial market is said to be *arbitrage-free* if there is no self-financing trading strategy that does not require the injection of capital at initiation, while it leads to a non-negative gain with probability 1 and to a positive gain with a positive probability. The financial market is said to be *complete* if any contingent claim admits a replicating strategy. A precise definition of these notions requires a careful specification of the set of admissible investment strategies. For a detailed description of these concepts, we refer e.g. to Karatzas & Shreve (1998).

Recall that a probability measure \mathbb{Q} defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T})$ is said to be an *equivalent martingale measure* (or a risk-neutral measure) for the financial market described above if \mathbb{Q} is equivalent to \mathbb{P} and any gains process, discounted at the risk-free rate, is a martingale with respect to \mathbb{Q} . In particular, under the risk-neutral measure \mathbb{Q} we have that

$$\begin{aligned} S(0) &= e^{-rt} \mathbb{E}[G(t)] \\ &= e^{-rt} \mathbb{E}[S(t)] + \mathbb{E} \left[\int_0^t e^{-ru} dD(u) \right]. \end{aligned} \quad (10)$$

In this expression and also hereafter, expectations have to be interpreted as expectations under the \mathbb{Q} -measure. Furthermore, the notation $F_{S(t)}$ will be used to denote the cdf of the stock price $S(t)$ under \mathbb{Q} .

Remark 5 Consider the r.v.'s X and Y , and let \mathbb{Q} be a probability measure, which is equivalent to underlying probability \mathbb{P} . Then it holds:

$$X = Y \text{ } \mathbb{P} - \text{a.s. if, and only if, } X = Y \mathbb{Q} - \text{a.s.}$$

From the First Fundamental Theorem of Asset Pricing we know that the no-arbitrage condition is ‘essentially equivalent’ to the existence of an equivalent martingale measure \mathbb{Q} . For a detailed discussion of this fundamental theorem in mathematical finance, we refer to Delbaen & Schachermayer (2008). Consider a traded contingent claim H which has a pay-off of $H(S(t))$ at time t . If the risk-neutral pricing measure in the market is given by \mathbb{Q} , then the price of $H(S(t))$ at time s , where $t \geq s \geq 0$, is equal to the conditional expected value of its discounted pay-off at time t :

$$\text{Price of } H(S(t)) \text{ at time } s = e^{-r(t-s)} \mathbb{E}[H(S(t)) \mid \mathcal{F}_s], \quad (11)$$

where the expectation is taken with respect to \mathbb{Q} . Hereafter, we will assume that there exists a risk-neutral measure \mathbb{Q} , and hence that the market is free of arbitrage.

In the sequel of this paper, we always assume that the stock S pays no dividends in $[0, T]$. Our results remain valid in the more general context where some of the stocks are paying dividends, provided the earned dividends are reinvested in the stock.

4 A market with a single non-dividend paying risky asset

In this subsection, we consider the financial market defined in Section 3 and with a single non-dividend paying risky asset, a stock let's say. The current price of the stock is given by $S(0)$ and its price process is given by the stochastic process $\{S(t) \mid 0 \leq t \leq T\}$. We assume that the market is arbitrage-free in the sense that prices of contingent claims are determined via the discounted expected pay-off, where expectations are taken under some equivalent martingale measure \mathbb{Q} .

Assume that $0 \leq t_1 < t_2 < \dots < t_n \leq T$. Let us also assume that the risk-neutral distributions of $S(t_1), S(t_2), \dots, S(t_n)$ are known. A natural question which arises now is whether an artificial market where the price vector $(S(t_1), S(t_2), \dots, S(t_n))$ is comonotonic is reachable, in the sense that it is arbitrage-free. Assuming that the marginal pricing distributions are fixed, this question can be restated as follows: does there exist an arbitrage-free market model, where the copula of $(S(t_1), S(t_2), \dots, S(t_n))$ in the \mathbb{Q} -world is the comonotonic copula?

As the measure \mathbb{P} and any equivalent martingale measure agree on sure events, almost sure relationships in the \mathbb{P} -world immediately translate in almost sure relationships in the pricing world and vice versa. Taking into account Definition 2, we can conclude that comonotonicity of $(S(t_1), S(t_2), \dots, S(t_n))$ in the risk-neutral world is equivalent to comonotonicity of this vector in the \mathbb{P} -world; see e.g. Dhaene et al. (2013).

In the following subsection, we derive conditions under which it is possible to have a comonotonic vector $(S(t_1), S(t_2), \dots, S(t_n))$, without violating the no-arbitrage assumption. We have a closer look at the Black & Scholes model in Example 13.

4.1 An arbitrage-free and comonotonic market

In order to be able to answer the question on reachability of the comonotonic market situation, we first have to prove the following lemma.

Lemma 6 *Consider the general arbitrage-free financial market with a single non-dividend paying stock and with risk-neutral measure \mathbb{Q} . In case there exists a functional relationship between the stock prices $S(s)$ and $S(t)$ at respective dates s and t with $s \leq t$, in the sense that*

$$S(t) = f(S(s)), \tag{12}$$

for some, not necessarily known, function f , then this relationship must be linear:

$$S(t) = S(s) e^{r(t-s)}. \tag{13}$$

Proof. We know that for the non-dividend paying stock S , we have that

$$S(s) = e^{-r(t-s)} \mathbb{E}[S(t) \mid \mathcal{F}_s].$$

Taking into account the relation (12), this expression can be transformed into

$$S(s) = e^{-r(t-s)}S(t),$$

which is equivalent to (13). ■

Relation (13) is a statement in the \mathbb{P} -world. Because \mathbb{P} and \mathbb{Q} are equivalent measures, it also holds in the \mathbb{Q} world.

Remark 7 The existence of a risk-neutral measure \mathbb{Q} is essential for the proof of Lemma 6. However, we can replace this assumption by the slightly weaker assumption that the market is arbitrage-free, provided we assume additionally that the function f is known at time s . In order to prove Lemma 6 under these adapted assumptions, we follow a reasoning ex absurdo. Therefore, suppose that

$$\mathbb{P} [S(t) \neq S(s) e^{r(t-s)}] > 0, \tag{14}$$

We will show that (14) leads to an arbitrage opportunity. Therefore, we set up the following self-financing trading strategy without initial capital injection. At time s , we observe the stock price $S(s)$ and compare $f(S(s))$ with $S(s)e^{r(t-s)}$. If $f(S(s)) > e^{r(t-s)}S(s)$, we borrow an amount $S(s)$ from the bank and buy one unit of the stock. At time t , we close our positions by selling the stock and repaying the debt $e^{r(t-s)}S(s)$ to the bank. In this case, the gain of the strategy equals

$$G = S(t) - e^{r(t-s)}S(s).$$

On the other hand, if at time s , it turns out that $f(S(s)) \leq e^{r(t-s)}S(s)$, we sell short one unit of the asset at time s and deposit the capital $S(s)$ on the bank account. At time t , we close our positions. In this case, the gain of the strategy is given by

$$G = e^{r(t-s)}S(s) - S(t).$$

Taking into account (14) we find that

$$\begin{aligned} \mathbb{P} [G \geq 0] &= \mathbb{P} [G \geq 0, f(S(s)) > e^{r(t-s)}S(s)] + \mathbb{P} [G \geq 0, f(S(s)) \leq e^{r(t-s)}S(s)] \\ &= \mathbb{P} [f(S(s)) > e^{r(t-s)}S(s)] + \mathbb{P} [f(S(s)) \leq e^{r(t-s)}S(s)] = 1. \end{aligned}$$

Similarly, we have that

$$\begin{aligned} \mathbb{P} [G > 0] &= \mathbb{P} [G > 0, f(S(s)) > e^{r(t-s)}S(s)] + \mathbb{P} [G > 0, f(S(s)) \leq e^{r(t-s)}S(s)] \\ &= \mathbb{P} [f(S(s)) > e^{r(t-s)}S(s)] + \mathbb{P} [f(S(s)) < e^{r(t-s)}S(s)] \\ &= \mathbb{P} [f(S(s)) \neq S(s) e^{r(t-s)}] > 0. \end{aligned}$$

We have proven that (14) leads to an arbitrage opportunity. Therefore, we can conclude that (13) must hold. ▽

4.2 The continuous case

Lemma 6 allows us to characterize comonotonicity of the vector $(S(t_1), S(t_2), \dots, S(t_n))$ of stock prices in an arbitrage-free market. Notice that $S(t_1)$ has a continuous cdf under \mathbb{P} if, and only if, it has a continuous cdf under \mathbb{Q} .

Theorem 8 *Consider the general arbitrage-free financial market with a single non-dividend-paying stock and with risk-neutral measure \mathbb{Q} . Assuming that $F_{S(t_1)}$ is a continuous cdf, the following statements are equivalent:*

(1)

$(S(t_1), S(t_2), \dots, S(t_n))$ is comonotonic.

(2) For any couple $(S(t_1), S(t_j))$, the following \mathbb{P} -a.s. relationship holds:

$$S(t_j) = S(t_1) e^{r(t_j - t_1)}, \quad j = 1, 2, \dots, n. \quad (15)$$

Proof. (1) Let us first assume that $(S(t_1), S(t_2), \dots, S(t_n))$ is a comonotonic random vector. As $F_{S(t_1)}$ is continuous, it follows from Theorem 4 that there exist non-decreasing functions f_i , $i = 2, 3, \dots, n$, such that

$$(S(t_1), S(t_2), \dots, S(t_n)) = (S(t_1), f_2(S(t_1)), \dots, f_n(S(t_1))).$$

The linear relationships (15) follow then from Lemma 6.

(2) The conditions (15) can be rewritten as

$$(S(t_1), S(t_2), \dots, S(t_n)) = (S(t_1), S(t_1) e^{r(t_2 - t_1)}, \dots, S(t_1) e^{r(t_n - t_1)}).$$

This implies that $(S(t_1), S(t_2), \dots, S(t_n))$ is comonotonic. ■

In Chen et al. (2015), the authors prove that strict comonotonicity of the price vector $(S(t_1), S(t_2), \dots, S(t_n))$ is equivalent with (15), while Albrecher et al. (2008) prove Theorem 8 for the case where all marginal cdf's $F_{S(t_i)}$, $i = 1, 2, \dots, n$ are continuous. Notice that Theorem 8 is directly applicable to many Lévy market models, as in most of these models $S(t_1)$ has a continuous cdf.

An essential assumption in Theorem 8 is the existence of a risk-neutral pricing measure \mathbb{Q} . Notice however, that the theorem remains to hold under the somewhat weaker assumption that the market is arbitrage-free, provided we assume additionally that the risk-neutral pricing distributions $F_{S(t_i)}$, $i = 1, 2, \dots, n$ of the individual stocks are known. This result follows from Remark 7 and the proof of Theorem 4.

Returning to the question that we considered in the beginning of this section, we can conclude from Theorem 8 that in case the marginal risk-neutral distributions $F_{S(t_i)}$, $i = 1, 2, \dots, n$, are given and $F_{S(t_1)}$ is a continuous cdf, it will in general be impossible to find an arbitrage-free market where the corresponding dependence structure is characterized by the comonotonic copula, unless these marginals cdf's are such that they fulfill the conditions (15). In the following example we show that the equivalence of (1) and (2) in Theorem 8 will not necessarily remain to hold in case $S(t_1)$ has a discrete cdf.

Example 9 Consider the well-known Cox-Ross-Rubinstein binomial model in the time interval $[0, 2]$. In this model, there is a bank account with constant interest intensity r and a non-dividend paying stock of which we observe the prices $S(0), S(1)$ and $S(2)$ at respective times $t = 0, 1, 2$. The stock price $S(0)$ at time 0 is given. Moreover, the stock price dynamics follow from

$$S(t+1) = S(t) \times Z(t+1), \quad t = 0, 1,$$

where $Z(1)$ and $Z(2)$ are i.i.d. r.v.'s which can only take positive values d and u , with $0 < d < u$, such that

$$\mathbb{P}[Z(t+1) = u] = 1 - \mathbb{P}[Z(t+1) = d] = p,$$

for some $0 < p < 1$. The couple $(S(1), S(2))$ is comonotonic since it has a comonotonic support. A necessary and sufficient condition for this model to be arbitrage-free is that

$$d \leq e^r \leq u, \quad (16)$$

or equivalently, that there exists an equivalent martingale measure \mathbb{Q} ; see e.g. Björk (1998). The equivalent martingale measure is determined by

$$\mathbb{Q}[Z(t+1) = u] = 1 - \mathbb{Q}[Z(t+1) = d] = \frac{e^r - d}{u - d}. \quad (17)$$

Any choice of d, u and r fulfilling the condition (16) leads to a discrete arbitrage-free market where $(S(1), S(2))$ is comonotonic, although the a.s. linear relationship (15) is violated. ∇

4.3 A generalization

From Theorem 8 we find that with given marginals, a comonotonic and arbitrage-free market is in general not attainable when $F_{S(t_1)}$ is continuous. In this section we generalize Theorem 8 by relaxing the assumption that the stock price $S(t_1)$ at time t_1 has a continuous cdf. We first introduce the notion of an isolated mass point of (the distribution of) a random variable.

Definition 10 *A r.v. X is said to have an isolated mass point x under the measure \mathbb{P} if $\mathbb{P}[X = x] > 0$ and there exists an $\varepsilon > 0$ such that $\mathbb{P}[X \in (x - \varepsilon, x) \cup (x, x + \varepsilon)] = 0$.*

We say that the r.v. X has no isolated mass under the measure \mathbb{P} if there are no isolated mass points, i.e. if for each $x \in \mathbb{R}$ with $\mathbb{P}[X = x] > 0$ and for every $\varepsilon > 0$, one has that $\mathbb{P}[X \in (x - \varepsilon, x) \cup (x, x + \varepsilon)] > 0$.

Notice that an isolated mass point of X under the measure \mathbb{P} remains an isolated mass point under an equivalent martingale measure, and vice versa. Also, X has no isolated mass under \mathbb{P} if, and only if, it has no isolated mass under any equivalent martingale measure \mathbb{Q} .

Theorem 11 *Consider the arbitrage-free market with a single non-dividend paying stock. Assume additionally that for any isolated mass point x of $S(t_1)$ and for any $j = 2, 3, \dots, n$, we have that*

$$\mathbb{P} [S(t_1) = x, S(t_j) < xe^{r(t_j-t_1)}] \times \mathbb{P} [S(t_1) = x, S(t_j) > xe^{r(t_j-t_1)}] = 0. \quad (18)$$

Then the following statements are equivalent:

(1)

$(S(t_1), S(t_2), \dots, S(t_n))$ is comonotonic.

(2) For any $j = 1, 2, \dots, n$, it holds

$$S(t_j) = S(t_1)e^{r(t_j-t_1)}. \quad (19)$$

A proof of Theorem 11 can be found in the appendix. In Example 9, we showed that in the Cox-Ross-Rubinstein binomial model in the time interval $[0, 2]$, the vector $(S(t_1), S(t_2))$ is comonotonic, without violating the no-arbitrage condition. The set of possible outcomes for $S(t_1)$ is given by $\{S(0)d, S(0)u\}$ and each point in this set is an isolated mass point. However, one can verify that condition (18) is not satisfied, implying that $(S(t_1), S(t_2))$ can be comonotonic without having the almost sure relationship (19).

Corollary 1 *Consider the arbitrage-free market with a single non-dividend paying stock. Assume additionally that $S(t_1)$ has no isolated mass. Then statements (1) and (2) from Theorem 11 are equivalent.*

As a special case from Corollary 1 we find that statements (1) and (2) from Theorem 11 are equivalent in case $S(t_1)$ has a continuous cdf, implying that Theorem 8 is a special case of Theorem 11.

Example 12 (An arbitrage-free market with comonotonic modification) Let the cdf $F_{S(t_1)}$ be such that there exists an isolated mass point x of $S(t_1)$, and

$$\begin{aligned} p_1 &= \mathbb{P} [S(t_2) < xe^{r(t_2-t_1)}] > 0, \\ p_2 &= \mathbb{P} [S(t_2) > xe^{r(t_2-t_1)}] > 0. \end{aligned}$$

Then one can construct an arbitrage-free market with comonotonic asset prices $(S(t_1), S(t_2))$ such that the almost sure relationship (19) does not hold. Indeed, we have that

$$q = 1 - p_1 - p_2 = \mathbb{P} [S(t_2) = xe^{r(t_2-t_1)}].$$

Let $m = \mathbb{P} [S(t_1) = x]$, and define F_1 and F_2 as follows:

$$\begin{aligned} F_1(y) &= \mathbb{P} [S(t_1) \leq y \mid S(t_1) \leq x], \\ F_2(y) &= \mathbb{P} [S(t_1) \leq y \mid S(t_1) \geq x]. \end{aligned}$$

The conditional cdf's G_1 and G_2 are defined as

$$\begin{aligned} G_1(y) &= \mathbb{P} [S(t_2) \leq y \mid S(t_2) < xe^{r(t_2-t_1)}], \\ G_2(y) &= \mathbb{P} [S(t_2) \leq y \mid S(t_2) > xe^{r(t_2-t_1)}]. \end{aligned}$$

Then we construct the cdf of \underline{Z} with

$$\begin{aligned} \mathbb{P} [\underline{Z} = (x, xe^{r(t_2-t_1)})] &= mq, \\ \mathbb{P} [\underline{Z} = (F_1^{-1}(U), G_1^{-1}(U))] &= p_1(1 - mq), \\ \mathbb{P} [\underline{Z} = (F_2^{-1}(U), G_2^{-1}(U))] &= p_2(1 - mq). \end{aligned}$$

The random vector \underline{Z} has marginal distributions identical to the distributions of $S(t_1)$ and $S(t_2)$, respectively. Furthermore, the market with comonotonic asset prices (Z_1, Z_2) is arbitrage-free, due to the fact that the probabilities

$$\mathbb{P} [\underline{Z} < (x, xe^{r(t_2-t_1)})] \quad \text{and} \quad \mathbb{P} [\underline{Z} > (x, xe^{r(t_2-t_1)})]$$

are positive. We also find that in this comonotonic and arbitrage-free market the almost sure relationship (19) does not hold. Let $\underline{Z} = (Z(t_1), Z(t_2))$. We set $Z(t) = Z(t_1)e^{r(t-t_1)}$ if $0 \leq t < t_1$, and $Z(t) = Z(t_2)e^{r(t-t_2)}$ if $t_2 \leq t \leq T$. We use the natural filtration generated by the process $Z(t)$, where \mathbb{P} -null sets are added to each sigma-field of the filtration. Then the constructed market is arbitrage-free, the random vector \underline{Z} is comonotonic, and the proportionality condition (19) does not hold for t_1 and t_2 . ∇

In case the stock price process $\{S(t) \mid t \geq 0\}$ can be described by the exponential of a Lévy process with Brownian part, the cdf $F_{S(t)}$ is continuous for any t and, consequently, comonotonicity of $(S(t_1), S(t_2), \dots, S(t_n))$ is, in general, not attainable in an arbitrage-free market. More details about the use of Lévy processes in finance can be found in Schoutens (2003).

Example 13 (An arbitrage-free market without comonotonic modification)

Consider a financial market with a constant and deterministic risk-free interest $r \geq 0$ and a single non-dividend paying stock. Suppose that the only information that is available about the random vector $(S(t_1), S(t_2), \dots, S(t_n))$ is that its risk-neutral marginal distributions follow from

$$\ln \frac{S(t_j)}{S(0)} \stackrel{\mathbb{Q}-d}{=} N \left(\left(r - \frac{1}{2}\sigma^2 \right) t_j, \sigma^2 t_j \right), \quad j = 1, 2, \dots, n, \quad (20)$$

with $\sigma > 0$. Here, $\stackrel{\mathbb{Q}-d}{=}$ denotes an equality in distribution under the \mathbb{Q} -measure. This information about the market may be observed e.g. via traded European option prices. This information is feasible in the sense that there exists an arbitrage-free market model where the risk-neutral distributions of the $S(t_j)$ are given by (20). Indeed, consider the Black & Scholes model where under \mathbb{P} on has that

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dB(t), \quad \text{for } t \geq 0,$$

where $\mu \geq 0$ and $\{B(t) \mid t \geq 0\}$ is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This probability space is equipped with the filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ of \mathcal{F} which records the ‘past behavior’ of the multivariate Brownian motion. This means that for $0 \leq t \leq T$, \mathcal{F}_t denotes the σ -algebra generated by $\{B(s) \mid 0 \leq s \leq t\}$, which we also denote by $\sigma\{B(s) \mid 0 \leq s \leq t\}$, completed by events of zero \mathbb{P} -probability. It is well-known that the Black & Scholes model is arbitrage free and complete. Under the unique martingale measure \mathbb{Q} , we have that

$$\ln \frac{S(t)}{S(0)} \stackrel{\mathbb{Q}-d}{=} N \left(\left(r - \frac{1}{2} \sigma^2 \right) t, \sigma^2 t \right).$$

This implies that the risk-neutral distributions of $\ln \frac{S(t_j)}{S(0)}$ are indeed given by (20). One can easily verify that in case the risk-neutral distributions $F_{S(t_i)}$ follow from (20), then for any $j \geq 2$, it holds that

$$S(t_j) \stackrel{\mathbb{Q}-d}{\neq} S(t_1) e^{r(t_j - t_1)}, \quad j = 2, \dots, n, \quad (21)$$

where the notation $\stackrel{\mathbb{Q}-d}{\neq}$ is used to indicate that $S(t_j)$ and $S(t_1) e^{r(t_j - t_1)}$ have different risk-neutral distributions. The inequalities (21) immediately imply that (15) does not hold. From Theorem 8 we can conclude that in case the risk-neutral distributions of the stock prices $S(t_j)$ follow from (20), then it is impossible to find an arbitrage-free market where the vector $(S(t_1), S(t_2), \dots, S(t_n))$ is comonotonic. ∇

5 A multiple asset market

In this section, we consider a financial market consisting of n stocks, labeled from 1 to n , and a bank account. The individual stocks can be described as in Section 3. The stocks may be dividend-paying or not. The prices of these stocks at a fixed future time T are given by the r.v.’s $S_i(T)$, $i = 1, 2, \dots, n$. Furthermore, the market is assumed to be arbitrage-free in the sense that prices are determined via some equivalent martingale measure \mathbb{Q} . In order to avoid unnecessary overloading of the notations, hereafter we will omit the fixed time index T when no confusion is possible. Hence, the random vector $(S_1(T), S_2(T), \dots, S_n(T))$ will be denoted as (S_1, S_2, \dots, S_n) if no confusion is possible.

Assume the only information available about the random behavior of the future prices of the different assets at a fixed time T is the set of marginal risk-neutral pricing distributions F_{S_i} , $i = 1, 2, \dots, n$. The question arises whether this limited information always allows for a market where the vector (S_1, S_2, \dots, S_n) is comonotonic. Notice that comonotonicity under \mathbb{P} is equivalent to comonotonicity under the equivalent martingale measure. The answer to our question is ‘yes’. Indeed, if we set

$$S_i = F_{S_i}^{-1}(U), \quad i = 1, 2, \dots, n,$$

where $U \sim \text{Uniform}(0, 1)$ and

$$S_i(0) = e^{-rT} \mathbb{E}[S_i], \quad i = 1, 2, \dots, n,$$

then the market with discrete times $t = 0, T$ and asset prices $S_i(t)$, $i = 1, 2, \dots, n$, the underlying probability is a martingale measure. A proof of this result is also given in Theorem 3.1 of Hobson et al. (2005). A related result can also be found in Proposition 3.6 of Tavin (2015).

5.1 The multivariate Black & Scholes model

Consider the financial market described in the previous section and assume that the price dynamics of stock i under \mathbb{P} follow from

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{P}}{=} \mu_i dt + \sigma_i dB_i(t), \quad t \geq 0 \text{ and } i = 1, 2, \dots, n, \quad (22)$$

where $\mu_i \geq 0$ is the drift and $\sigma_i > 0$ the volatility of stock i . The parameters r, μ_i and σ_i are assumed to be deterministic and constant over time. Furthermore, $\{\underline{B}(t) \mid t \geq 0\}$ is an n -dimensional correlated Brownian motion process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This means that the stochastic processes $\{B_i(t) \mid t \geq 0\}$ are standard Brownian motions and the dependence structure of $\{\underline{B}(t) \mid t \geq 0\}$ is captured by

$$\text{Cov}[\sigma_i B_i(t), \sigma_j B_j(t+s)] = \sigma_{ij} t, \text{ for any } t, s \geq 0, \quad (23)$$

with parameters σ_{ij} , $i, j = 1, 2, \dots, n$, which are also assumed to be deterministic and constant over time.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is equipped with the natural filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ of \mathcal{F} , which records the ‘past behavior’ of the multivariate Brownian motion process. The set \mathcal{F}_t is the σ -algebra $\sigma\{\underline{B}(s), s \leq t\}$ completed by events of zero probability. In the sequel, we will refer to this model as the *Multivariate Black & Scholes model*.

Hereafter, the notation $\underline{1}$ is used for the $n \times 1$ matrix with all entries equal to 1. Furthermore, we denote the $n \times 1$ matrix of the drifts by $\underline{\mu}$, while the $n \times n$ matrix with entries σ_{ij} is denoted by $\underline{\Sigma}$. The range $R(\underline{\Sigma})$ of $\underline{\Sigma}$ is defined by

$$R(\underline{\Sigma}) = \{\underline{y} \mid \underline{y} = \underline{\Sigma} \underline{x} \text{ for some } n \times 1 \text{ - matrix } \underline{x}\}. \quad (24)$$

The Black & Scholes model (22) is not necessarily arbitrage-free. A precise definition of the notions of arbitrage and completeness within the Black & Scholes model is considered e.g. in Dhaene et al. (2014) and the references therein. Concerning the absence of arbitrage in the Black & Scholes model (22), the following result holds.

Theorem 14 *In the Black & Scholes model (22), the following statements are equivalent:*

- (1) *The model is arbitrage-free.*
- (2) *There exists an equivalent martingale measure.*
- (3) *There exists a unique equivalent martingale measure.*
- (4) $\underline{\mu} - r\underline{1} \in R(\underline{\Sigma})$.

When the Black & Scholes model (22) is arbitrage-free, the risk-neutral pricing dynamics of the stocks follows from

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{Q}}{=} rdt + \sigma_i dB_i(t), \quad i = 1, 2, \dots, n. \quad (25)$$

In particular, we find that in the \mathbb{Q} -world, $\ln \frac{S_i(t)}{S_i(0)}$ follows a normal distribution with mean $(r - \frac{1}{2}\sigma_i^2)t$ and standard deviation $\sigma_i\sqrt{t}$. Hence we have that

$$\ln \frac{S_i(t)}{S_i(0)} \stackrel{\mathbb{Q}-d}{=} N \left(\left(r_i - \frac{1}{2}\sigma_i^2 \right) t, \sigma_i^2 t \right). \quad (26)$$

As the matrix $\underline{\Sigma}$ is necessary positive semidefinite, we have that $\underline{\Sigma}$ is positive definite if and only if $R(\underline{\Sigma}) = \mathbb{R}^n$. Combining this observation with Theorem 14, we find the following corollary.

Corollary 2 *A sufficient (but not necessary) condition for the Black & Scholes model (22) to be arbitrage-free is that $\underline{\Sigma}$ is positive definite.*

Hereafter, we consider some special cases of the Black & Scholes model (22). We will say that the Black & Scholes market is comonotonic if the stock price vector (S_1, S_2, \dots, S_n) is comonotonic.

Example 15 (a comonotonic B&S market which is not arbitrage-free) Consider a two dimensional Black & Scholes model (22)

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{P}}{=} \mu_i dt + \sigma dB(t), \quad i = 1, 2.$$

From expression (31) in Example 17, we find that this model can only be arbitrage-free if

$$\mu_1 = \mu_2.$$

Assume for the moment that $\mu_1 > \mu_2$ and $S_1(0) = S_2(0)$. Consider the strategy of buying one unit of the first stock and sell one unit of the second stock. This position does not require any capital input at time $t = 0$. However, because the first asset will grow with a faster speed than the second one, we have that

$$S_1(t) > S_2(t), \text{ for } t > 0,$$

which means that the investor will make a sure profit, with an initial investment of 0. Here, the B&S market is comonotonic, but not arbitrage free.

▽

Example 16 (an arbitrage-free B&S market which is not comonotonic) Consider the following special case of the Black & Scholes model (22):

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{P}}{=} \mu_i dt + \sigma dB_i(t), \quad i = 1, 2, \dots, n, \quad (27)$$

In this case we find that

$$\sigma_{ij} = \sigma^2 \text{Cov}[B_i(1), B_j(1)].$$

Let us now assume that the matrix $\underline{\Sigma}$ is positive definite. From our considerations above, we find that the model (27) is arbitrage-free and complete. Its risk-neutral pricing dynamics follows from

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{Q}}{=} rt + \sigma dB_i(t), \quad i = 1, 2, \dots, n. \quad (28)$$

In particular, we find that

$$\ln \frac{S_i(t)}{S_i(0)} \stackrel{\mathbb{Q}-d}{=} N\left(\left(r - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right). \quad (29)$$

We can conclude that we have found an arbitrage-free B&S market, where the price vector (S_1, S_2, \dots, S_n) is not comonotonic.

▽

Example 17 (arbitrage-free and comonotonic B&S market) Consider the following special case of the Black & Scholes model (22):

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{P}}{=} \mu_i dt + \sigma_i dB(t), \quad i = 1, 2, \dots, n, \quad (30)$$

where the stochasticity of all stock prices is captured by a single standard Brownian motion process $\{B(t) \mid t \geq 0\}$. In this model, we have that for each time t , the price vector $(S_1(t), \dots, S_n(t))$ is comonotonic because each $S_i(t)$ is an increasing function of the same r.v. $B(t)$. The process $\{B(t) \mid t \geq 0\}$ plays the role of a random index which influences the behavior of stock prices in the market. In this case we find that

$$\sigma_{ij} = \sigma_i \sigma_j.$$

The range of $\underline{\Sigma}$ is given by

$$R(\underline{\Sigma}) = \left\{ \left(\sum_{j=1}^n \sigma_j x_j \right) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix} \middle| \underline{x} \in \mathbb{R}^n \right\}.$$

Taking into account that $\left\{ \left(\sum_{j=1}^n \sigma_j x_j \right) \mid \underline{x} \in \mathbb{R}^n \right\} = \mathbb{R}$, we can rewrite the previous expression as

$$R(\underline{\Sigma}) = \left\{ \lambda \left(\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{array} \right) \mid \lambda \in \mathbb{R} \right\}.$$

From Theorem 14 we find that the Black & Scholes model (22) is arbitrage-free if, and only if, there exists a common real number c such that

$$\frac{\mu_i - r}{\sigma_i} = c, \quad i = 1, 2, \dots, n. \quad (31)$$

The risk-neutral price dynamics of the model (30) follow from:

$$\frac{dS_i(t)}{S_i(t)} \stackrel{\mathbb{Q}}{=} r dt + \sigma_i dB(t), \quad i = 1, 2, \dots, n. \quad (32)$$

In particular, we find that

$$\ln \frac{S_i(t)}{S_i(0)} \stackrel{\mathbb{Q}-d}{=} N \left(\left(r - \frac{1}{2} \sigma_i^2 \right) t, \sigma_i^2 t \right). \quad (33)$$

We can conclude that under the condition (31), the B&S market is comonotonic and arbitrage-free.

▽

Until here, we assumed that the risk-neutral pricing distributions of the individual stocks are given. Let us now consider the situation where the real-world pricing distributions of the individual stocks are given. In the following example, we show that in this case it is not always possible to find an arbitrage-free market with comonotonic price vector (X_1, X_2, \dots, X_n) .

Example 18 (comonotonic B&S market which is not arbitrage-free) Suppose that the real-world distributions of the price at time T of the n assets follow from

$$\ln \frac{S_i}{S_i(0)} \stackrel{\mathbb{P}-d}{=} N \left(\left(\mu_i - \frac{1}{2} \sigma^2 \right) T, \sigma^2 T \right), \quad i = 1, 2, \dots, n, \quad (34)$$

with

$$\mu_1 < \mu_2 < \dots < \mu_n. \quad (35)$$

Here, we write $\stackrel{\mathbb{P}-d}{=}$ to denote an equality in distribution under the measure \mathbb{P} . These marginal distributions may be observed in the sense that they are equal to the real world pricing distributions of the arbitrage-free Black & Scholes model that we considered in Example 16.

In this case it is impossible to construct an arbitrage-free market model where the random

vector (S_1, S_2, \dots, S_n) is comonotonic. Indeed, in case this vector was comonotonic, we would find that

$$(S_1, S_2, \dots, S_n) \stackrel{\mathbb{P}^{-d}}{=} \left(S_1(0) e^{(\mu_1 - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\Phi^{-1}(U)}, \dots, S_n(0) e^{(\mu_n - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}\Phi^{-1}(U)} \right), \quad (36)$$

which holds in the real world. An arbitrage opportunity arises by buying e.g. $\frac{S_1(0)}{S_2(0)}$ units of the the second asset and selling one unit of the first asset at time 0, while closing this position at time T .

▽

6 Conclusions

In this note we investigated the relationship between arbitrage-free markets and comonotonicity of traded asset prices. We first investigated the meaning of comonotonicity of the prices $(S(t_1), \dots, S(t_n))$ of a single asset at consecutive dates t_1, t_2, \dots, t_n in an arbitrage-free market with a single risky asset and a risk-free bank account with interest intensity r . We proved that the vector of subsequent prices $(S(t_1), \dots, S(t_n))$ is comonotonic only under the a.s. linear relationship

$$S(t_j) = S(t_i) e^{(r-\nu)(t_j-t_i)}, i, j = 1, 2, \dots, n,$$

provided $S(t_1)$ has a continuous cdf, or more generally, provided the distribution of $S(t_1)$ has no isolated mass.

Next, we considered an arbitrage-free financial market consisting of n traded assets and a risk-free bank account. The prices of these assets at a fixed future time T are given by the r.v.'s $S_i(T)$, $i = 1, 2, \dots, n$. We considered situations under which the risk-neutral pricing distributions (or the real world distributions) of the $S_i(T)$ are given and investigated whether there exists an arbitrage-free model with the same marginal distributions, under which the asset price vector $(S_1(T), S_2(T), \dots, S_n(T))$ is comonotonic.

Acknowledgement 1 *The authors acknowledge the financial support of the Onderzoeksfonds KU Leuven (GOA/13/002: Management of Financial and Actuarial Risks: Modeling, Regulation, Incentives and Market Effects) and the Research Fund Flanders (FWO G.OC38.17N: Measuring diversity and systematic risk from derivative market data and mapping to real-world probability statements). They are grateful to Prof. Yuliya Mishura (Ukraine) and Dr. Xinliang Chen (Belgium) for fruitful discussions.*

References

Albrecher, H., Dhaene, J., Goovaerts, M. & Schoutens, W. (2005), ‘Static hedging of Asian options under Lévy models: the comonotonicity approach’, *Journal of Derivatives* **12**(3), 63–72.

- Albrecher, H., Mayer, P. A. & Schoutens, W. (2008), ‘General lower bounds for arithmetic asian option prices’, *Applied Mathematical Finance* **15**(2), 123–149.
- Barrieu, P. M. & Scandolo, G. (2015), ‘Assessing financial model risk’, *European Journal of Operational Research* **242**(2), 546–556.
- Chen, X., Deelstra, G., Dhaene, J., Linders, D. & Vanmaele, M. (2015), ‘On an optimization problem related to static super-replicating strategies’, *Journal of Computational and Applied Mathematics* **278**, 213–230.
- Chen, X., Deelstra, G., Dhaene, J. & Vanmaele, M. (2008), ‘Static super-replicating strategies for a class of exotic options’, *Insurance: Mathematics & Economics* **42**(3), 1067–1085.
- Cheung, K. (2007), ‘Characterizations of conditional comonotonicity’, *Journal of Applied Probability* **44**(3), 607–617.
- Cheung, K. C. (2009), ‘Upper comonotonicity’, *Insurance: Mathematics & Economics* **45**(1), 35–40.
- Cheung, K. C. (2010), ‘Characterizing a comonotonic random vector by the distribution of the sum of its components’, *Insurance: Mathematics & Economics* **47**(2), 130 – 136.
- Deelstra, G., Dhaene, J. & Vanmaele, M. (2011), An overview of comonotonicity and its applications in finance and insurance, in B. Oksendal & G. Nunno, eds, ‘Advanced Mathematical Methods for Finance’, Springer Berlin Heidelberg, pp. 155–179.
- Deelstra, G., Diallo, I. & Vanmaele, M. (2008), ‘Bounds for Asian basket options’, *J. Comput. Appl. Math.* **218**, 215–228.
- Deelstra, G., Liinev, J. & Vanmaele, M. (2004), ‘Pricing of arithmetic basket options by conditioning’, *Insurance: Mathematics and Economics* **34**(1), 55–77.
- Delbaen, F. & Schachermayer, W. (2008), *The mathematics of arbitrage*, Springer. correcter second printing, pp. 373.
- Denneberg, D. (1997), *Non-Additive measure and integral, 2nd edition*, Kluwer Academic Publishers. pp. 184.
- Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. & Vyncke, D. (2002a), ‘The concept of comonotonicity in actuarial science and finance: theory’, *Insurance: Mathematics & Economics* **31**(1), 3–33.
- Dhaene, J., Denuit, M., Goovaerts, M., Kaas, R. & Vyncke, D. (2002b), ‘The concept of comonotonicity in actuarial science and finance: applications’, *Insurance: Mathematics & Economics* **31**(2), 133–161.
- Dhaene, J., Dony, J., Forsys, M. B., Linders, D. & Schoutens, W. (2012a), Fix - the fear index: Measuring market fear, in ‘Topics in Numerical Methods for Finance, Cummins M. et al. (eds.). Springer Proceedings in Mathematics and Statistics’.

- Dhaene, J., Kukush, A. & Linders, D. (2014), ‘The multivariate Black & Scholes market: conditions for completeness and no-arbitrage’, *Theory of Probability and Mathematical Statistics* **88**, 85–98.
- Dhaene, J., Kukush, A., Luciano, E., Schoutens, W. & Stassen, B. (2013), ‘On the (in-)dependence between financial and actuarial risks’, *Insurance: Mathematics and Economics* **52**(3), 522 – 531.
- Dhaene, J., Linders, D., Schoutens, W. & Vyncke, D. (2012b), ‘The herd behavior index: A new measure for the implied degree of co-movement in stock markets’, *Insurance: Mathematics and Economics* **50**(3), 357–370.
- Dhaene, J., Linders, D., Schoutens, W. & Vyncke, D. (2014a), ‘A multivariate dependence measure for aggregating risks’, *Journal of Computational and Applied Mathematics* **263**(0), 78 – 87.
- Hobson, D., Laurence, P. & Wang, T. (2005), ‘Static-arbitrage upper bounds for the prices of basket options’, *Quantitative Finance* **5**(4), 329–342.
- Kaas, R., Dhaene, J. & Goovaerts, M. J. (2000), ‘Upper and lower bounds for sums of random variables’, *Insurance: Mathematics and Economics* **27**(2), 151–168.
- Karatzas, I. & Shreve, S. E. (1998), *Methods of Mathematical Finance*, Springer-Verlag, New York.
- Kerkhof, J., Melenberg, B. & Schumacher, H. (2010), ‘Model risk and capital reserves’, *Journal of Banking & Finance* **34**(1), 267 – 279.
- Linders, D., Dhaene, J., Hounnon, H. & Vanmaele, M. (2012), Index options: a model-free approach, Research report AFI-1265 FEB, Leuven: KU Leuven - Faculty of Business and Economics.
- Linders, D. & Stassen, B. (2016), ‘The multivariate Variance Gamma model: basket option pricing and calibration’, *Quantitative Finance* **16**(4), 555–572.
- Schoutens, W. (2003), *Lévy processes in Finance: Pricing Financial Derivatives*, Wiley.
- Simon, S., Goovaerts, M. & Dhaene, J. (2000), ‘An easy computable upper bound for the price of an arithmetic Asian option’, *Insurance: Mathematics & Economics* **26**(2), 175–183.
- Tavin, B. (2015), ‘Detection of arbitrage in a market with multi-asset derivatives and known risk-neutral marginals’, *Journal of Banking and Finance* **53**, 158–178.

Appendix: A.1. Proof of Theorem 11

Proof. We give a proof for the case where $n = 2$. The proof for the case $n > 2$ is straightforward.

The proof of (2) \Rightarrow (1) is trivial.

Let us now prove (1) \Rightarrow (2).

Assume that (19) does not hold. Let $x_i, i = 1, 2, \dots, n$, be all the isolated mass points of the r.v. $S(t_1)$, and $y_j, j = 1, 2, \dots, n$, be all the other points where $F_{S(t_1)}$ is discontinuous.

We introduce the following notations:

$$\begin{aligned}\Omega_d^1 &= \{\omega \in \Omega \mid S(t_1)(\omega) \in \{x_i, i = 1, 2, \dots, n\}\}, \\ \Omega_d^2 &= \{\omega \in \Omega \mid S(t_1)(\omega) \in \{y_j, j = 1, 2, \dots, n\}\}, \\ \Omega_c &= \Omega \setminus (\Omega_d^1 \cup \Omega_d^2).\end{aligned}$$

1. In a first step, we prove that

$$S(t_2)(\omega) = S(t_1)(\omega)e^{r(t_2-t_1)}, \quad \text{for all } \omega \in \Omega_c, \text{ a.s.} \quad (\text{A1})$$

Here we suppose that $\mathbb{P}[\Omega_c] > 0$ (otherwise it is clear that (A1) holds true).

Define the restricted probability space $(\Omega_c, \mathcal{F} \cap \Omega_c, \mathbb{P}/\mathbb{P}(\Omega_c))$ and restricted r.v.'s

$$\begin{aligned}S^r(t_i)(\omega) &= S(t_i)(\omega), \quad \omega \in \Omega_c, i = 1, 2, \\ S^r(0) &= S(0).\end{aligned}$$

Then $S^r(t_1)$ has a continuous cdf, the vector $(S^r(t_1), S^r(t_2))$ is still comonotonic, and the market with a single risky stock $S^r(t), t = 0, t_1, t_2$, is still arbitrage-free. Then, by Theorem 8, we have that

$$S^r(t_2)(\omega) \stackrel{\text{a.s.}}{=} S^r(t_1)(\omega)e^{r(t_2-t_1)},$$

and (A1) follows.

2. Let us now additionally assume that there exists an i such that

$$\mathbb{P}[S(t_1) = x_i, S(t_2) > x_i e^{r(t_2-t_1)}] > 0. \quad (\text{A2})$$

Because x_i is an isolated mass point of $S(t_1)$ and (18) holds with $S = x_i$, we have that

$$\mathbb{P}[S(t_1) = x_i, S(t_2) < x_i e^{r(t_2-t_1)}] = 0. \quad (\text{A3})$$

Let us now consider the following strategy. At the moment t_1 the investor observes $S(t_1)$. If $S(t_1) \neq x_i$, he does nothing, if $S(t_1) = x_i$, he recognizes the situation (A2), (A3), borrows the amount x_i from the bank, immediately buys one unit of the stock and at the moment t_2 he sells it, returns his debt, and receives $S(t_2) - x_i e^{r(t_2-t_1)}$. Overall, $S(t_2) - x_i e^{r(t_2-t_1)} \geq 0$, a.s., and $\mathbb{P}[S(t_2) - x_i e^{r(t_2-t_1)} > 0] > 0$, and we have an arbitrage strategy. This contradicts the condition that the market is arbitrage-free.

3. In a similar way, an arbitrage strategy can be constructed under the condition that there exists an x_i with $\mathbb{P} [S(t_1) = x_i, S(t_2) < x_i e^{r(t_2-t_1)}] > 0$.

4. Thus, from steps 2 and 3 we can conclude that

$$S(t_2)(\omega) = S(t_1)(\omega)e^{r(t_2-t_1)}, \text{ for all } \omega \in \Omega_d^1, \text{ a.s.} \quad (\text{A4})$$

5. Now, suppose that (A1) and (A4) hold and there exists j with

$$\mathbb{P} [S(t_1) = y_j, S(t_2) > y_j e^{r(t_2-t_1)}] > 0, \quad (\text{A5})$$

$$\mathbb{P} [S(t_1) = y_j, S(t_2) < y_j e^{r(t_2-t_1)}] > 0. \quad (\text{A6})$$

Then there exists an $\varepsilon \in (0, y_j)$ with

$$\mathbb{P} [S(t_1) = y_j, S(t_2) > (y_j + \varepsilon)e^{r(t_2-t_1)}] > 0, \quad (\text{A7})$$

$$\mathbb{P} [S(t_1) = y_j, S(t_2) < (y_j - \varepsilon)e^{r(t_2-t_1)}] > 0. \quad (\text{A8})$$

Because y_j is not an isolated mass point of $S(t_1)$, we can assume that

$$\mathbb{P} [S(t_1) \in (y_j, y_j + \varepsilon)] > 0.$$

(We will exploit (A7), and in case $\mathbb{P} [S(t_1) \in (y_j - \varepsilon, y_j)] > 0$ we could exploit (A8)). Now, (A7) and condition (1) of Theorem 11 imply that

$$\mathbb{P} [S(t_1) \in (y_j, y_j + \varepsilon), S(t_2) > (y_j + \varepsilon)e^{r(t_2-t_1)}] > 0,$$

$$\mathbb{P} [S(t_1) \in (y_j, y_j + \varepsilon), S(t_2) \leq (y_j + \varepsilon)e^{r(t_2-t_1)}] = 0.$$

Then due to (A1), there exists $z \in (y_j, y_j + \varepsilon)$ which satisfies

$$\mathbb{P} [S(t_1) = z, S(t_2) > z e^{r(t_2-t_1)}] > 0,$$

$$\mathbb{P} [S(t_1) = z, S(t_2) < z e^{r(t_2-t_1)}] = 0.$$

These relations are similar to (A2) and (A3), and part 2 of the proof shows that we get a contradiction.

6. For a fixed j , the case (A5) with $\mathbb{P} [S(t_1) = y_j, S(t_2) < y_j e^{r(t_2-t_1)}] = 0$ is impossible due to the reasoning from part 2 of the proof. In a similar way the case (A6) with $\mathbb{P} [S(t_1) = y_j, S(t_2) > y_j e^{r(t_2-t_1)}] = 0$ is impossible as well.

7. Suppose that (A1) and (A4) hold. Then parts 5 and 6 of the proof imply that $S(t_2)(\omega) = S(t_1)(\omega)e^{r(t_2-t_1)}$, for all $\omega \in \Omega_d^2$, a.s. Finally, (A1), (A4) and the latter relation imply (19). ■