A GENERALIZED STOCHASTIC METHOD FOR ESTIMATING THE CHARACTERISTICS OF POTENTIAL CONFLICTS OF A CONTROLLED AIR TRAFFIC

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A generalized stochastic method is presented for evaluating conflict characteristics such as conflict probability, collision probability, integral estimate of conflict probability on the near-collision time interval, and mean time to a predicted conflict. Equations are obtained for finding these conflict characteristics with regard for the stochastic nature and time correlation of the deviation from a planned controlled-flight trajectory.

Keywords: *air traffic control, norm of safe separation of aircraft, conflict forecast, probability of conflict, stochastic disturbance model, diffusion process.*

INTRODUCTION

Development and constant perfection of air traffic management (ATM), implementation of the concept of a global (CNS/ATM) (communication, navigation, surveillance and ATM) system, new concepts of ATM such as cooperative ATM, free flight, etc. impose new stringent requirements on collision avoidance systems and on the techniques for estimating air traffic safety level. Of the variety of the well-known stochastic methods of estimating conflict probability [1], we may emphasize the method of estimating conflict probability [2] and the method of estimating collision risk [3].

The former method [2] is intended for medium-term prediction of potentially conflicting situations for a period of 15 to 20 minutes. The method determines an analytical expression for estimating the conflict probability for two aircraft; it is based on a priori information on the variance of longitudinal, transverse, and vertical flight path errors. These errors are considered independent of each other and normally distributed. All of the uncertainty of the future position of two aircraft is assigned to one of them as a summary covariance matrix. Let the domain of uncertainty be an ellipsoid in space, the other aircraft be described by a sphere of a radius equal to the safe clearance. While moving relative to the first aircraft, this sphere forms a cylinder. Geometrically, the probability of a conflict is defined as the probability measure of intersection of the uncertainty domain and the cylinder.

The second method [3], which is based on the concept of collision risk, is preferable for short-term (5 to 10 minutes) conflict detection. This method is a generalization of the well-known Reich method [4]. The relative motion (separation) of aircraft is a random process, with known statistical characteristics of the position and velocity, described by a stochastic differential equation. Theoretically, the collision probability is determined as the probability of the first intersection of this process with a forbidden domain with dimensions commensurable with aircraft. However, since the exact mathematical solution is extremely complex, the method is simplified. Multiple entries into the forbidden domain are admitted, and the function of the intersection intensity and then probability of intersection are determined. As a result, the collision risk, rather than the collision probability, is determined, which is the upper bound of the collision probability.

Flight control is a highly automated process. Nevertheless, an aircraft may significantly deviate from the assigned flight path, which is a random process. The main factors causing deviations and, therefore, uncertainty of the future position

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of the aircraft are wind, navigational errors, flight path measurement errors, inadequate mathematical models that describe the motion of the aircraft, errors of the path computation algorithms, control errors, and human errors.

In the paper, the authors present a generalized stochastic approach for the estimation of the characteristics of potentially conflict situations: conflict probability (i.e., the probability that the norm of safe separation (or collision) of aircraft is violated), integral estimate of the conflict probability, an mean time to a predicted conflict with regard for stochastic disturbances and correlation time dependence of the generally multidimensional process of deviation from assigned flight path for controlled air traffic.

GENERAL FORMULATION OF THE PROBLEM

In estimating conflict probability and collision risk, stochastic models of aircraft motion are used, where deviations from the given flight path are described by Wiener processes [5]. For example, experimental data on actual flight paths [6] are close to the results of simulation based on the hypothesis of the Wiener nature of the random process of deviation from the given flight path.

Let the motion of each jth aircraft in the state space be described by the multidimensional stochastic differential equation

$$d\mathbf{X}_j(t) = f(\mathbf{X}_j(t), t)dt + g(\mathbf{X}_j(t), t)d\mathbf{W}_j(t), \quad \mathbf{X}_j(t) \in \mathfrak{R}^N, \tag{1}$$

where $\mathbf{X}_j(t)$ is the vector of states and $\mathbf{W}_j(t)$ is a multidimensional Wiener process.

Let the combined vector of states $\mathbf{X}(t) \in \mathfrak{R}^{2N}$ for two aircraft be

$$\mathbf{X}(t) = [\mathbf{X}_1(t)^T, \mathbf{X}_2(t)^T]^T;$$

and the stochastic differential equation for it be

$$d\mathbf{X}(t) = F(\mathbf{X}(t), t)dt + G(\mathbf{X}(t), t)d\mathbf{W}(t), \quad \mathbf{X}(t) \in \mathfrak{R}^{2N}, \tag{2}$$

where $\mathbf{W}(t) = [\mathbf{W}_1(t)^T, \mathbf{W}_2(t)^T]^T$.

Denote the process of separation of two aircraft (the vector of distance between the aircraft) by $\mathbf{r}(t)$ for $\mathbf{r}(t) \in \mathfrak{R}^n$ $(n < 2N)$; it may be obtained as a continuous mapping from $\mathfrak{R}^{2N}$ into $\mathfrak{R}^n$ by the operator $\mathfrak{F}$,

$$\mathbf{r}(t) = \mathfrak{F}\mathbf{X}(t).$$

We will now determine the conflict domain. Let $D_r = S^c$ determine the conflict domain for $\mathbf{r}(t)$, where S^c is the complement of S to the whole space $\mathfrak{R}^n$, and $S \subset \mathfrak{R}^n$ is an open set with boundary $\partial S \subset S^c$. If D_x determines the conflict domain for $\mathbf{X}(t)$, then $D_x = \mathfrak{F}^{-1}(D_r)$.

MATHEMATICAL MODELS OF RANDOM DEVIATIONS OF AIRCRAFT AND THEIR RELATIVE MOTION

In considering peculiarities of flight control, assume that the motion of each aircraft (1) is described in a partial (local) (generally three-dimensional) coordinate system with the axes x_1, x_2, x_3 oriented along the required path x_1, in the transverse direction x_2, and vertically upwards x_3, respectively; the motion in one coordinate being independent of the motion in the other coordinates.

To derive mathematical model of deviations from a prescribed flight path, we will use the Ornstein–Uhlenbeck random process [8], owing to some of its properties. The Ornstein–Uhlenbeck stable process is a unique random process, which is stationary, Markovian, Gaussian, and has continuous trajectories. These properties are most adequate to the real controlled steady motion of an aircraft influenced by random disturbing factors.

Let us use the Ornstein–Uhlenbeck random process to describe the deviation of an aircraft from the flight velocity $\tilde{v}_i(t)$ for each ith coordinate:

$$d\tilde{v}_i = -\alpha_i \tilde{v}_i dt + \sigma_i dW_i, \tag{3}$$

where $\alpha_i > 0$, $\sigma_i > 0$, and $W_i = W_i(t)$ is a standard Wiener process. The processes $\{W_i(t)\}$ are mutually independent for different coordinates.

Let us represent each component of the motion process $\mathbf{X}_i(t) = [x_i(t), v_i(t)]^T$ along the ith coordinate in a three-dimensional coordinate system $i = \overline{1,3}$ as

$$\mathbf{X}_i(t) = \overline{\mathbf{X}}_i(t) + \widetilde{\mathbf{X}}_i(t). \tag{4}$$

Here $\overline{\mathbf{X}}_i(t) = [\overline{x}_i(t), \overline{v}_i(t)]^T$ are the parameters of planned flight path, where $\overline{v}_i(t) = v_{i\,\mathrm{plan}}$ is a given flight velocity, $\overline{x}_i(t) = x_i(0) + v_{i\,\mathrm{plan}}\, t$ is the nominal (planned) change of coordinate; and $\widetilde{\mathbf{X}}_i(t) = [\widetilde{x}_i(t), \widetilde{v}_i(t)]^T$ are deviations from the given (planned) parameters of the flight path.

The deviation for the ith coordinate with regard for the model (3) is

$$d\widetilde{\mathbf{X}}_i(t) = \widetilde{v}_i(t)\begin{bmatrix} 1 \\ -\alpha_i \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma_i \end{bmatrix} dW_i(t),$$

and for the complete vector of deviations

$$\widetilde{\mathbf{X}}(t) = [\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3, \widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3]^T \tag{5}$$

we have

$$d\widetilde{\mathbf{X}}(t) = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \widetilde{\mathbf{X}}(t) dt + \begin{bmatrix} E \\ F \end{bmatrix} d\mathbf{W}(t), \tag{6}$$

where $A, B, \ldots, F \in \mathfrak{R}^{3\times 3}$; $A, C, E = 0$; $B = I_3$ is a unit 3×3 matrix; $D = -\mathrm{diag}[\alpha_1, \alpha_2, \alpha_3]$; $F = \mathrm{diag}[\sigma_1, \sigma_2, \sigma_3]$; and $\mathbf{W} = [W_1, W_2, W_3]^T$.

Let us define the process of separation $\mathbf{r}(t)$ of two aircraft in a unified coordinate system (in a special case, it can be one of the local (partial) coordinate systems).

Let us denote the complete vector of states for the jth aircraft in the partial coordinate system by

$$\mathbf{X}_j(t) = [x_{j,1}, x_{j,2}, x_{j,3}, v_{j,1}, v_{j,2}, v_{j,3}]^T \tag{7}$$

and introduce a combined vector of states for two aircraft, satisfying Eq. (2):

$$\mathbf{X} = [\mathbf{X}_1^T, \mathbf{X}_2^T]^T = [x_{1,1}, x_{1,2}, x_{1,3}, v_{1,1}, v_{1,2}, v_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, v_{2,1}, v_{2,2}, v_{2,3}]^T. \tag{8}$$

Let us also introduce, for the jth aircraft, a complete vector of states that defines the path parameters in a unified coordinate system but with respect to the origin of the corresponding local system $\mathbf{X}_{0j}$ whose coordinates are known,

$$\mathbf{X}'_j(t) = [x'_{j,1}, x'_{j,2}, x'_{j,3}, v'_{j,1}, v'_{j,2}, v'_{j,3}]^T. \tag{9}$$

Let us define the operator of transformation U_j from the local (7) to the displaced unified coordinate system (9) such that

$$\mathbf{X}'_j = U_j(\mathbf{X}_j). \tag{10}$$

Given the definitions (7), (9), and (10), we will express the process of separation as

$$\mathbf{r}(t) = \mathbf{r}_{12} + P_x\mathbf{X}'_2(t) - P_x\mathbf{X}'_1(t) = \mathbf{r}_{12} + P_x(U_2(\mathbf{X}_2) - U_1(\mathbf{X}_1)), \tag{11}$$

where $\mathbf{r}_{12}$ is the vector of the distances between the local coordinate systems of the first and second aircraft with the known coordinates $\mathbf{X}_{01}$ and $\mathbf{X}_{02}$ in the unified coordinate system, i.e., $\mathbf{r}_{12} = \overrightarrow{M_{01}M_{02}}$ where the point M_{0j} has the coordinates $\mathbf{X}_{0j}$; and P_x is the operator of projection of the state vector onto the subspace of coordinate components,

$$P_x\mathbf{X}'_j = [x'_{j,1}, x'_{j,2}, x'_{j,3}]^T. \tag{12}$$

The relation between the process (11) and the combined multidimensional vector of deviations

$$\tilde{\mathbf{X}} = [\tilde{\mathbf{X}}_1^T, \tilde{\mathbf{X}}_2^T]^T$$

$$= [\tilde{x}_{1,1}, \tilde{x}_{1,2}, \tilde{x}_{1,3}, \tilde{v}_{1,1}, \tilde{v}_{1,2}, \tilde{v}_{1,3}, \tilde{x}_{2,1}, \tilde{x}_{2,2}, \tilde{x}_{2,3}, \tilde{v}_{2,1}, \tilde{v}_{2,2}, \tilde{v}_{2,3}]^T \quad (13)$$

is established in view of definitions (4) and (12) by the relation

$$\mathbf{r}(t) = \mathbf{r}_{12} + P_x\mathbf{X}'_2 - P_x\mathbf{X}'_1 = \mathbf{r}_{12} + P_x(U_2(\overline{\mathbf{X}}_2 + \tilde{\mathbf{X}}_2) - U_1(\overline{\mathbf{X}}_1 + \tilde{\mathbf{X}}_1))$$

$$= \mathbf{r}_{12} + P_x(U_2(\overline{\mathbf{X}}_2) - U_1(\overline{\mathbf{X}}_1)) + P_x(U_2(\tilde{\mathbf{X}}_2) - U_1(\tilde{\mathbf{X}}_1)), \quad (14)$$

in this case, for the jth aircraft for complete vectors (5) and (7)

$$\mathbf{X}'_j = U_j(\overline{\mathbf{X}}_j + \tilde{\mathbf{X}}_j) = \overline{\mathbf{X}}'_j + \tilde{\mathbf{X}}'_j,$$

and for the combined vectors (8) and (13)

$$\mathbf{X}' = [\mathbf{X}_1^{\prime T}, \mathbf{X}_2^{\prime T}]^T, \ \tilde{\mathbf{X}}' = [\tilde{\mathbf{X}}_1^{\prime T}, \tilde{\mathbf{X}}_2^{\prime T}]^T.$$

CHARACTERISTICS OF A POTENTIAL CONFLICT

The main characteristics of a predictable conflict situation may be considered the probability of violating the norm of safe separation of aircraft (or the collision probability), the relative average duration of a conflict, and mean time up to the first violation of safe separation (or collision).

Denote by A the set of events corresponding to the case where the distance between the aircraft (14) becomes equal or less than a given norm of safe separation d:

$$A = \{\mathbf{X}' \in \mathfrak{R}^{12} : ||\mathbf{r}_{12} + P_x(\mathbf{X}'_2 - \mathbf{X}'_1)|| \le d\}$$

$$= \{\mathbf{X}' \in \mathfrak{R}^{12} : ||\mathbf{r}_{12} + P_x(\overline{\mathbf{X}}'_2 - \overline{\mathbf{X}}'_1) + P_x(\tilde{\mathbf{X}}'_2 - \tilde{\mathbf{X}}'_1)|| \le d\} \quad (15)$$

$$= \{\mathbf{X}' \in \mathfrak{R}^{12} : ||\mathbf{r}(t)|| \le d\}.$$

Let us define the probabilistic characteristics of potentially conflict situations.

1. Let us define the probability of violating the norm of safe separation of aircraft on the time interval $[0, T]$ as

$$P_c = P\{\exists\, t \in [0, T] : ||\mathbf{r}(t)|| \le d\} = 1 - P, \quad (16)$$

where P is the probability that safe separation will not be violated on the interval $[0, T]$. Note that if the value of d is determined by the physical dimensions of an aircraft, then the above conditions mean aircraft collision.

Let safe separation occur at the initial time $t = 0$, i.e., $||\mathbf{r}(0)|| = ||\mathbf{r}_{12}|| > d$.

2. Let us define the relative average duration of a conflict on the near-collision interval $[t_0, T]$ by using the Lebesgue measure of set (mes)

$$P_\Sigma = \frac{E\,\mathrm{mes}\,\{t \in [t_0, T] : ||\mathbf{r}(t)|| \le d\}}{T - t_0}. \quad (17)$$

3. Let us define mean time up to the first conflict as

$$T_c = E\tau, \quad (18)$$

where $\tau = \min\{t > 0 : \ ||\mathbf{r}(t)|| = d\}$.

THE EQUATION FOR ESTIMATING THE COLLISION PROBABILITY

The problem of revealing a conflict is similar to the well-known problem of achieving boundaries by a multidimensional Markovian process, for which a partial differential equation of parabolic type [7] is defined in a general form. The solution of this equation yields the probability that the process achieves the boundary.

Let us derive the equations for estimating the conflict probability and additional characteristics of a conflict situation, using another approach. For the collision probability introduced, let us derive a partial differential equation using an the infinitesimal operator $\mathcal{A}$ of the corresponding diffusion process that acts on the function $v(t,\mathbf{x})$ from the class $C^2(\mathfrak{R}^d)$ by the law [11]

$$\mathcal{A}v(\mathbf{x}) = \sum_{i=1}^{d} b_i(\mathbf{x})\frac{\partial v}{\partial x_i} + \frac{1}{2}\sum_{i,k=1}^{d} a_{ik}\frac{\partial^2 v}{\partial x_i \partial x_k}. \tag{19}$$

Let us use the following Feinman–Katz theorem [9, p. 366].

THEOREM 1. Let $v(t,\mathbf{x}):[0,T]\times\mathfrak{R}^d \to \mathfrak{R}$ be a continuous function that belongs to the class $C^{1,2}([0,T]\times\mathfrak{R}^d)$ (i.e., $v\in C^{1,2}((0,T)\times\mathfrak{R}^d)$ and the corresponding partial derivatives admit continuous continuation on $[0,T]\times\mathfrak{R}^d$). Let $v(t,\mathbf{x})$ satisfy the Cauchy problem

$$-\frac{\partial v}{\partial t} + k(\mathbf{x})v = \mathcal{A}v \text{ on } [0,T]\times\mathfrak{R}^d,\ v(T,\mathbf{x}) = 1,\ \ \mathbf{x}\in\mathfrak{R}^d, \tag{20}$$

where $k:\mathfrak{R}^d \to [0,\infty)$ is continuous, and the growth condition $\max_{0\le t\le T} |v(t,\mathbf{x})| \le M(1+||\mathbf{x}||^{2\mu})$, $\mathbf{x}\in\mathfrak{R}^d$, is satisfied with the some $M>0$, $\mu\ge 1$. Then we can present

$$v(t,\mathbf{x}) = E^{t,\mathbf{x}} e^{-\int_t^T k(\mathbf{x}_\theta)d\theta}, \qquad t\in[0,T],\ \ \mathbf{x}\in\mathfrak{R}^d, \tag{21}$$

where $\mathbf{x}_\theta$, $\theta\in[0,T]$, is the diffusion process in $\mathfrak{R}^d$ with the infinitesimal operator $\mathcal{A}$ (19), and $E^{t,\mathbf{x}}$ means the expectation provided that $\mathbf{x}_t = \mathbf{x}$. (In particular, this solution of the Cauchy problem is unique.)

Let us formulate a theorem for the infinitesimal operator of the diffusion process.

THEOREM 2. Let us consider a linear stochastic differential equation

$$\mathbf{X}_s^{t,\mathbf{X}} = \mathbf{X} + \int_t^s \mathbf{b}(\mathbf{X}_\theta^{t,\mathbf{X}})d\theta + \int_t^s \sigma d\mathbf{W}_\theta,\ \ t\le s<\infty. \tag{22}$$

Here $\mathbf{W}_\theta$ is Brownian motion in $\mathfrak{R}^r$, σ is a $d\times r$ matrix, $\mathbf{b}=\mathbf{b}(\mathbf{X})$ is an affine function in $\mathbf{X}\in\mathfrak{R}^r$ with the values in $\mathfrak{R}^d$: $\mathbf{b}(\mathbf{X}) = \mathbf{b}_0 + b_1\mathbf{X}$, $\mathbf{b}_0\in\mathfrak{R}^d$, $b_1\in\mathfrak{R}^{r\times d}$; all the vectors being column vectors.

Let $\sigma\sigma^T = (a_{ik})_{i,k=1}^d$ and $\mathcal{A}$ be the infinitesimal operator corresponding to the given stochastic differential equation (22). Let also $\mathbf{b}_0 = (b_0^i)_{i=1}^d$ and $b_1 = (b_1^{ij})_{i,j=1}^d$. Then

$$(\mathcal{A}g)(\mathbf{X}) = \sum_{i=1}^{d}\left(b_0^i + \sum_{j=1}^{d} b_1^{ij}(X_j)\right)\frac{\partial g}{\partial X_i} + \frac{1}{2}\sum_{i,k=1}^{d} a_{ik}\frac{\partial^2 g}{\partial X_i \partial X_k}. \tag{23}$$

Theorem 2 gives the structure of the infinitesimal operator and allows us to determine its coefficients for the given diffusion process.

Let us now introduce a function similar to the functional $v(t,\mathbf{x})$ (21) from Theorem 1,

$$P_\lambda(t,\mathbf{X}') = P(t,\mathbf{X}') = E^{t,\mathbf{X}'} e^{-\lambda\int_t^T I(\mathbf{Z}(\theta)\in \mathrm{A})d\theta} \tag{24}$$

for $\lambda>0$, $t\in[0,T]$, $\mathbf{X}'\in\mathfrak{R}^{12}$, where

$$\mathbf{Z}(\theta)=\begin{bmatrix}\widetilde{\mathbf{X}}'_1(\theta)\\ \widetilde{\mathbf{X}}'_2(\theta)\end{bmatrix},\quad \theta\ge 0, \tag{25}$$

$$A=\{\mathbf{X}'\in \mathrm{R}^{12}:||\mathbf{r}_{12}+P_x(\mathbf{X}'_2-\mathbf{X}'_1)||\le d\},\quad \mathbf{X}'=[\mathbf{X}_1^{'T},\ \mathbf{X}_2^{'T}]^T,$$

I is the indicator function, equal to unity if $\mathbf{Z}(\theta)\in A$ and equal to zero if $\mathbf{Z}(\theta)\notin A$.

Note that at the initial time $t=0$, the position of the aircraft is such that $\mathbf{Z}(0)\notin A$, i.e., no collision occurs. Let us consider the value of the function (24):

$$P_\lambda(0,\mathbf{X}')=E^{0,\mathbf{X}'}e^{-\lambda\int_0^T I(\mathbf{Z}(\theta)\in \mathrm{A})\,d\theta},\quad \text{where } \mathbf{X}'=\mathbf{Z}(0)\notin A.$$

As $\lambda\to+\infty$, by the Lebesgue theorem on dominated convergence [10]

$$P_\lambda(0,\mathbf{X}')\to P\left\{\int_0^T I(\mathbf{Z}(\theta)\in A)\,d\theta=0\right\}.$$

Let us fix the initial value $\mathbf{Z}(0)=\mathbf{X}'$. Since $\mathbf{Z}(\theta)$, $\theta\ge 0$, has continuous trajectories and the set A is closed,

$$\lim_{\lambda\to\infty} P_\lambda(0,\mathbf{X}')=P\{\mathbf{Z}(\theta)\notin A\quad \forall\,\theta\in[0,T]\}=P,$$

where P is defined in (16).

Similarly, for all $t\in[0,T]$,

$$\lim_{\lambda\to\infty} P_\lambda(t,\mathbf{X}')=P(t,\mathbf{X}')=P\{\mathbf{Z}(\theta)\notin A\ \ \forall\theta\in[t,T]\,|\,\mathbf{Z}(t)=\mathbf{X}'\}.$$

The last probability is conditional for $\mathbf{Z}(t)=\mathbf{X}'$.

The limiting relations obtained show that for large λ, the functions $P_\lambda(t,\mathbf{X}')$ are the smoothed approximations of the functions $P(t,\mathbf{X}')$.

We are interested in the value $P(0,\mathbf{X}')=P$, which determines the probability of not violating the norms of safety on the time interval $t\in[0,T]$.

According to Theorem 1, one may state that the function $P_\lambda(t,\mathbf{X}')$ defined by expression (24) satisfies Eq. (20)

$$-\frac{\partial P_\lambda}{\partial t}+\lambda I(\mathbf{X}'\in A)P_\lambda=\mathcal{A}P_\lambda,\quad t\in[0,T);$$
$$P_\lambda(T,\mathbf{X}')=1,\quad \mathbf{X}'\in\Re^{12}. \tag{26}$$

This statement is not strict since the function $k(\mathbf{X}')=\lambda I(\mathbf{X}'\in A)$ is discontinuous on the boundary $\partial A=\{\mathbf{X}'\in\Re^{12}:||\mathbf{r}_{12}+P_x(\mathbf{X}'_2-\mathbf{X}'_1)||=d\}$. However, Eq. (26) can be obtained from Theorem 1 for the function $\lambda I(\mathbf{X}'\in A)$ by approximating this function with continuous functions. We will substantiate this below.

In determining the collision probability, we are interested in the case where the initial conditions are such that $\mathbf{X}'\notin A$. Then Eq. (26) becomes

$$-\frac{\partial P_\lambda}{\partial t}=\mathcal{A}P_\lambda,\quad t\in[0,T),\quad \mathbf{X}'\in\Re^{12}\setminus A=A^c. \tag{27}$$

It is necessary to determine the limit $P(t,\mathbf{X}')$ of $P_\lambda(t,\mathbf{X}')$ as $\lambda\to+\infty$. Then we obtain from (27) the following equation for $P(t,\mathbf{X}')$:

$$-\frac{\partial P(t,\mathbf{X}')}{\partial t}=\mathcal{A}P(t,\mathbf{X}'),\quad t\in[0,T),\quad \mathbf{X}'\in A^c;$$

$$P(t,\mathbf{X}')=0,\quad \mathbf{X}'\in A^0=\{\mathbf{X}'\in\Re^{12}:||\mathbf{r}_{12}+P_x(\mathbf{X}'_2-\mathbf{X}'_1)||\le d\},$$

since for $\mathbf{X}'\in A^0$ we have $P\left\{\int_t^T I(\mathbf{Z}(\theta)\in A)\,d\theta=0\,|\,\mathbf{Z}(t)=\mathbf{X}'\right\}=0$.

Since $P(t, \mathbf{X}')$ is continuous in $\mathbf{X}'$, we have

$$P(t, \mathbf{X}') = 0, \quad \mathbf{X}' \in \partial A = \{\mathbf{X}' \in \mathfrak{R}^{12} : ||\mathbf{r}_{12} + P_x(\mathbf{X}'_2 - \mathbf{X}'_1)|| = d\}.$$

Moreover,

$$P(T, \mathbf{X}') = 1, \quad \mathbf{X}' \in A^c.$$

Thus, the equation for the probability that aircraft do not violate the norm of their safe separation is

$$-\frac{\partial P(t, \mathbf{X}')}{\partial t} = \mathcal{A} P(t, \mathbf{X}'), \quad t \in [0, T), \quad \mathbf{X}' \in A^c; \tag{28}$$

$$P(T, \mathbf{X}') = 1, \ \ \mathbf{X}' \in A^c, \ \text{for} \ t \in [0, T) \quad \lim_{\substack{\mathbf{X}' \in A^c \\ \mathbf{X}' \to \mathbf{X}_0 \in \partial A}} P(t, \mathbf{X}') = 0.$$

The function $P(t, \mathbf{X}')$ is continuous, $0 \le P(t, \mathbf{X}') \le 1$, it is given on the set

$$D = \{(t, T) : t \in [0, T), \ \mathbf{X}' \in A^c \cup \partial A, \ \text{or} \ t = T, \ \mathbf{X}' \in A^c\}.$$

Thus, the problem of estimating the probability P_c of violating norms of safe separation of two aircraft on the time interval $t \in [t_0, T]$ is solved as follows: 1) determine the differential operator $\mathcal{A}$ with the structure (23) according to Theorem 2 for the process $\mathbf{Z}(\theta)$ that is defined in (25) and includes the deviations, reduced to a unified coordinate system, which satisfy Eq. (6) for each aircraft; 2) solve Eq. (28) by numerical methods; 3) determine the probability $P = P(0, \mathbf{X}')$ for $\mathbf{X}' = \mathbf{Z}(0)$; 4) finally determine the unknown probability $P_c = 1 - P$ according to (16).

INTEGRAL ESTIMATE OF THE COLLISION PROBABILITY

The above equations estimate the collisions probability for a given fixed time of prediction. Another important characteristic of the potential conflict is the integral estimate of the probability P_Σ of a conflict on the near-collision interval $[t_0, T]$, determined by expression (17). Let us obtain the equation for this probability and conditions of its solution.

Let us consider the expression for the function (24) and Eq. (26) which it satisfies.

For $t \in [0, T)$, $\mathbf{X}' \in A^0$ we have

$$-\frac{\partial P_\lambda}{\partial t} + \lambda I(\mathbf{X}' \in A) P_\lambda = \mathcal{A} P_\lambda$$

or

$$-\frac{\partial P_\lambda}{\partial t} + \lambda P_\lambda = \mathcal{A} P_\lambda, \quad P_0(t, \mathbf{X}') = 1, \quad P_\lambda(T, \mathbf{X}') = 1. \tag{29}$$

For $t \in [0, T)$, $\mathbf{X}' \in A^c$, we have

$$-\frac{\partial P_\lambda}{\partial t} = \mathcal{A} P_\lambda, \quad P_0(t, \mathbf{X}') = 1, \quad P_\lambda(T, \mathbf{X}') = 1. \tag{30}$$

The solution of the system of equations (29), (30) should be joined continuously on the boundary ∂A. We have

$$\frac{\partial}{\partial \lambda} P_\lambda(t, \mathbf{X}') = -E^{t, \mathbf{X}'} \int_t^T I(\mathbf{Z}(\theta) \in A) d\theta \, e^{-\lambda \int_t^T I(\mathbf{Z}(\theta) \in A) d\theta}, \tag{31}$$

$$\frac{\partial}{\partial \lambda} P_\lambda(t, \mathbf{X}') \Big|_{\lambda = 0, t = 0} = -E^{0, \mathbf{X}'} \int_0^T I(\mathbf{Z}(\theta) \in A) d\theta = -E^{0, \mathbf{X}'} \text{mes} \{\theta \in [0, T] : \mathbf{Z}(\theta) \in A\},$$

$$E^{0, \mathbf{X}'} \text{mes} \{\theta \in [0, T] : \mathbf{Z}(\theta) \in A\} = -\frac{\partial}{\partial \lambda} P_\lambda(t, \mathbf{X}') \Big|_{\lambda = 0, t = 0}.$$

The initial condition for the case considered is $\mathbf{X}' \notin A$.

Let us write the equation for (31). From (29) we have the following.

For $\mathbf{X}' \in A^0$,

$$-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \lambda} P_\lambda\right) + P_\lambda + \lambda \frac{\partial}{\partial \lambda} P_\lambda = \mathcal{A} \frac{\partial}{\partial \lambda} P_\lambda, \quad \frac{\partial}{\partial \lambda} P_\lambda(T, \mathbf{X}') = 0.$$

For $\mathbf{X}' \in A^c$, from (30) we obtain

$$-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \lambda} P_\lambda\right) = \mathcal{A} \frac{\partial}{\partial \lambda} P_\lambda, \quad \frac{\partial}{\partial \lambda} P_\lambda(T, \mathbf{X}') = 0.$$

Denote $\varphi(t, \mathbf{X}') = -\frac{\partial}{\partial \lambda} P_\lambda(t, \mathbf{X}')\big|_{\lambda = 0}$.

If $\lambda = 0$, we have:

(1) for $\mathbf{X}' \in A^0$,

$$-\frac{\partial}{\partial t}\varphi - 1 = \mathcal{A}\varphi, \quad t \in [0, T), \quad \varphi(T, \mathbf{X}') = 0; \tag{32}$$

(2) for $\mathbf{X}' \in A^c$,

$$-\frac{\partial}{\partial t}\varphi = \mathcal{A}\varphi, \quad t \in [0, T), \quad \varphi(T, \mathbf{X}') = 0; \tag{33}$$

in all cases, $0 \le \varphi(t, \mathbf{X}') \le T - t$.

The equations obtained should be solved by numerical methods, by joining the solution for $\mathbf{X}' \in \partial A$ so that it becomes continuous on $t \in [0, T]$, $\mathbf{X}' \in \mathfrak{R}^{12}$.

Given (31), we find

$$E^{0, \mathbf{X}'} \text{mes}\,\{\theta \in [0, T] : \mathbf{Z}(\theta) \notin A\} = T + \frac{\partial}{\partial \lambda} P_\lambda \Big|_{\substack{\lambda = 0, \\ t = 0}} = T - \varphi(0, \mathbf{X}').$$

Then

$$E^{0, \mathbf{X}'} \text{mes}\,\{t \in [t_0, T] : \mathbf{Z}(t) \notin A\} = E^{0, \mathbf{X}'} \text{mes}\,\{t \in [0, T] : \mathbf{Z}(t) \notin A\}$$

$$- E^{0, \mathbf{X}'} \text{mes}\,\{t \in [0, t_0] : \mathbf{Z}(t) \notin A\} = (1 - P_\Sigma)(T - t_0)$$

or

$$(1 - P_\Sigma)(T - t_0) = \varphi(0, \mathbf{X}')|_{T = t_0} - \varphi(0, \mathbf{X}')|_{T = T}.$$

Herefrom, we derive the unknown collision probability according to (17):

$$P_\Sigma = 1 - \frac{\varphi(0, \mathbf{X}')|_{T = t_0} - \varphi(0, \mathbf{X}')|_{T = T}}{T - t_0}. \tag{34}$$

ESTIMATING THE MEAN TIME TO A CONFLICT SITUATION

The mean time of possible aircraft collision $T_c = E\tau$, determined by (18), is an important characteristic of a potentially conflict situation.

If for a given simple event $\omega \in \Omega$, the event $\|\mathbf{r}(\omega)\| = d$ (violation of safety norms) never occurs, then we put $\tau(\omega) = +\infty$.

We may write $\tau = \max\{t \in [0, T] : \mathbf{Z}(\theta) \in A \ \ \forall\, \theta \in [0, t)\}$.

Let $0 < s < T$. Tail of the distribution function of the random variable τ:

$$P^{0, \mathbf{X}'}\{\tau > s\} = P^{0, \mathbf{X}'}\{\mathbf{Z}(\theta) \notin A \quad \forall\, \theta \in [0, s]\} = P(0, \mathbf{X}')|_{T = s},$$

where $\mathbf{X}' \notin A$ is the given initial condition.

The equation for this quantity has the form (28). For different s, it can be determined by numerical methods.

The estimate of the mean time $E^{0,\mathbf{X}'}\tau = \int_0^T P^{0,\mathbf{X}'}\{\tau > s\}ds$ can be found by numerical integration.

JUSTIFICATION THE CORRECTNESS OF APPLYING THEOREM 1

Let us approximate the function $\mathbf{X}' \mapsto I(\mathbf{X}' \in A)$ in (24) by the continuous function

$$g_\varepsilon(\mathbf{X}') = \begin{cases} 1, & \text{if } \mathbf{X}' \in A, \quad \rho(\mathbf{X}', \partial A) \ge \varepsilon; \\ 0, & \text{if } \mathbf{X}' \in A^c, \quad \rho(\mathbf{X}', \partial A) \ge \varepsilon; \\ \dfrac{1}{2} - \dfrac{\rho(\mathbf{X}', \partial A)}{2\varepsilon}, & \text{if } \mathbf{X}' \in A^c, \quad \rho(\mathbf{X}', \partial A) < \varepsilon; \\ \dfrac{1}{2} + \dfrac{\rho(\mathbf{X}', \partial A)}{2\varepsilon}, & \text{if } \mathbf{X}' \in A, \quad \rho(\mathbf{X}', \partial A) < \varepsilon. \end{cases} \tag{35}$$

Then we obtain $P_\lambda^\varepsilon(t, \mathbf{X}')$ instead of $P_\lambda(t, \mathbf{X}')$, $P_\lambda^\varepsilon(t, \mathbf{X}') = E^{t,\mathbf{X}'} e^{-\lambda \int_t^T g_\varepsilon(\mathbf{Z}(\theta))d\theta}$. Here $\rho(\mathbf{X}', \partial A) = \min_{\mathbf{Y} \in \partial A} ||\mathbf{X}' - \mathbf{Y}||$ is the distance from $\mathbf{X}'$ to ∂A, $\varepsilon < \dfrac{d}{\sqrt{2}}$, where d is the threshold from (15).

The function $P_\lambda^\varepsilon(t, \mathbf{X}')$ satisfies the equation

$$-\frac{\partial P_\lambda^\varepsilon}{\partial t} + \lambda g_\varepsilon(\mathbf{X}') P_\lambda^\varepsilon = \mathcal{A} P_\lambda^\varepsilon, \quad t \in [0, T), \quad \mathbf{X}' \in \mathfrak{R}^{12};$$

$$P_\lambda^\varepsilon(T, \mathbf{X}') = 1, \quad \mathbf{X}' \in \mathfrak{R}^{12}.$$

Note that

$$\rho(\mathbf{X}', \partial A) = \frac{|\,||\mathbf{r}_{12} + P_x(\mathbf{X}'_2 - \mathbf{X}'_1)|| - d|}{\sqrt{2}}, \quad \mathbf{X}' = \begin{bmatrix} \mathbf{X}'_1 \\ \mathbf{X}'_2 \end{bmatrix} \in \mathfrak{R}^{12}.$$

The limit of the function $P_\lambda^\varepsilon(t, \mathbf{X}')$, as $\lambda \to +\infty$, is determined by the expression

$$P_\infty^\varepsilon(t, \mathbf{X}') = E^{t,\mathbf{X}'} I\left(\int_t^T g_\varepsilon(\mathbf{Z}(\theta))d\theta = 0 \right) = P^{t,\mathbf{X}'}\left\{ \int_t^T g_\varepsilon(\mathbf{Z}(\theta))d\theta = 0 \right\},$$

$$P_\infty^\varepsilon(t, \mathbf{X}') = P^{t,\mathbf{X}'}\{g_\varepsilon(\mathbf{Z}(\theta)) = 0 \ \ \forall\, \theta \in [t, T]\}$$

$$= P^{t,\mathbf{X}'}\{\forall\, \theta \in [t, T]: \mathbf{Z}(\theta) \in A^c, \ \rho(\mathbf{Z}(\theta), \partial A) \ge \varepsilon\}$$

$$= P^{t,\mathbf{X}'}\{\forall\, \theta \in [t, T]: ||\mathbf{r}_{12} + P_x(\mathbf{X}'_2(\theta) - \mathbf{X}'_1(\theta))|| \ge d + \varepsilon\sqrt{2}\}.$$

If $d_1 = d + \varepsilon\sqrt{2}$, then for

$$A_1 = \left\{ \begin{bmatrix} \mathbf{X}'_1 \\ \mathbf{X}'_2 \end{bmatrix} \in \mathfrak{R}^{12} : ||\mathbf{r}_{12} + P_x(\mathbf{X}'_2 - \mathbf{X}'_1)|| \le d_1 \right\} \tag{36}$$

we have

$$\underset{\lambda\to+\infty}{P_\lambda^\varepsilon}(t,\mathbf{X}')=P^{t,\mathbf{X}'}\left\{\mathbf{Z}(\theta)\in A_1^c \quad \forall\,\theta\in[t,T]\right\},$$

where $A_1^c=\mathfrak{R}^{12}\setminus A_1$. Then

$$P^{t,\mathbf{X}'}\{\exists\,\theta\in[t,T]:\ \mathbf{Z}(\theta)\in A_1\}=1-\lim_{\lambda\to+\infty} P_\lambda^\varepsilon\ (t,\mathbf{X}').$$

The limit equation for $P_\infty^\varepsilon(t,\mathbf{X}')$ is

$$-\frac{\partial P_\infty^\varepsilon}{\partial t}=\mathcal{A}P_\infty^\varepsilon,\quad t\in[0,T),\quad \mathbf{X}'\in\mathfrak{R}^{12};$$

$$P_\infty(t,\mathbf{X}')=0,\ \ \mathbf{X}'\in A_1,\ t\in[0,T);$$

$$P_\infty(T,\mathbf{X}')=1,\ \ \mathbf{X}'\in A_1^c;$$

$$0\le P_\infty\le 1.$$

The function P_∞^ε is continuous on $D=([0,T)\times A_1)\cup([0,T]\times A_1^c)$. Actually, one should consider $P_\infty^\varepsilon(t,\mathbf{X}')$ only on the set D. Then

$$-\frac{\partial P_\infty^\varepsilon}{\partial t}=\mathcal{A}P_\infty^\varepsilon,\quad t\in[0,T),\quad \mathbf{X}'\in A_1^c,\ P_\infty^\varepsilon\in C(D);$$

$$P_\infty(t,\mathbf{X}')=0,\ \ \mathbf{X}'\in\partial A_1;$$

$$P_\infty(T,\mathbf{X}')=1,\ \ \mathbf{X}'\in A_1;$$

$$0\le P_\infty\le 1.$$

Thus, we have obtained for A_1 from (36) the same equations as for the conflict domain A from (15) for the functions $P_\lambda(t,\mathbf{X}')$ of the form (24). Therefore, we may eliminate ε, and, as before, consider the set A instead of A_1.

To substantiate the correctness of deriving the integral probability P_Σ in (34), let us apply approximation (35) in expressions (31):

$$-\frac{\partial}{\partial\lambda}P_\lambda^\varepsilon\bigg|_{\lambda=0}=E^{t,\mathbf{X}'}\int_t^T g_\varepsilon(\mathbf{Z}(\theta))\,d\theta.$$

This characteristic allows us to approximate the measure

$$E^{t,\mathbf{X}'}\,\mathrm{mes}\,\{\theta\in[0,T]:\mathbf{Z}(\theta)\in A\}.$$

Indeed,

$$|g_\varepsilon(\mathbf{Z}(\theta))-I(\mathbf{Z}(\theta)\in A)|\le I(\rho(\mathbf{Z}(\theta),\partial A)<\varepsilon,\ \ \theta\in[t,T]);$$

therefore,

$$\left|E^{t,\mathbf{X}'}\int_t^T g_\varepsilon(\mathbf{Z})d\theta-E^{t,\mathbf{X}'}\,\mathrm{mes}\,\{\theta\in[t,T]:\mathbf{Z}(\theta)\in A\}\right|$$

$$\le E^{t,\mathbf{X}'}\,\mathrm{mes}\,\{\theta\in[t,T]:\rho(\mathbf{Z}(\theta),\partial A)<\varepsilon\}$$

$$=E^{t,\mathbf{X}'}\,\mathrm{mes}\left\{\theta\in[t,T]:\ \frac{|\,\|\mathbf{r}_{12}+P_x(\mathbf{X}_2'-\mathbf{X}_1')\|-d|}{\sqrt{2}}<\varepsilon\right\}.$$

As $\varepsilon \to 0+$, this quantity tends to zero; therefore, as $E^{t,\mathbf{X}'} \text{mes}\{\theta \in [0,T]: \mathbf{Z}(\theta) \in A\}$, one may approximately use

$$\varphi^{\varepsilon}(t, \mathbf{X}') = E^{t,\mathbf{X}'} \int_t^T g_{\varepsilon}(\mathbf{Z}(\theta)) d\theta = -\frac{\partial}{\partial \lambda} P_{\lambda}^{\varepsilon} \Big|_{\lambda=0}.$$

Let us derive the equation for this quantity. From the equation for $P_{\lambda}^{\varepsilon}$ we have

$$-\frac{\partial}{\partial t}\left(\frac{\partial P_{\lambda}^{\varepsilon}}{\partial \lambda}\right) + g_{\varepsilon} P_{\lambda}^{\varepsilon} + \lambda g_{\varepsilon} \frac{\partial P_{\lambda}^{\varepsilon}}{\partial \lambda} = \mathcal{A} \frac{\partial P_{\lambda}^{\varepsilon}}{\partial \lambda},$$

$$-\frac{\partial}{\partial t}\left(-\frac{\partial P_{\lambda}^{\varepsilon}}{\partial \lambda}\right) - g_{\varepsilon} P_{\lambda}^{\varepsilon} - \lambda g_{\varepsilon} \frac{\partial P_{\lambda}^{\varepsilon}}{\partial \lambda} = \mathcal{A}\left(-\frac{\partial P_{\lambda}^{\varepsilon}}{\partial \lambda}\right).$$

Putting here $\lambda = 0$, we obtain with regard for $P_0^{\varepsilon} = 1$:

$$-\frac{\partial}{\partial t}\varphi^{\varepsilon} - g_{\varepsilon} = \mathcal{A}\varphi^{\varepsilon}, \quad t \in [0, T), \quad \mathbf{X}' \in \mathfrak{R}^{12};$$

$$\varphi^{\varepsilon}(T, \mathbf{X}') = 0, \quad \mathbf{X}' \in \mathfrak{R}^{12}, \quad 0 \le \varphi^{\varepsilon}(t, \mathbf{X}') \le T - t.$$

These equations approximate Eqs. (32) and (33).

Thus, the correctness of the application of Theorem 1 is proved.

CONCLUSIONS

The authors propose here a generalized stochastic method for estimating probabilistic characteristics of potentially conflict situations that may arise under certain flight conditions. Such characteristics are the probability of violating the norm of safe separation of aircraft or the probability of their collision on a near-collision interval, and also the integral criterion of the probability of violating safe separation (collision) and the mean time of predictable violation (collision). The problem is formulated with regard for the stochastic nature of disturbances that cause aircraft to deviate from prescribed flight paths, and correlation time dependences of a multidimensional, in the general case, process of deviation of a controlled air traffic in three-dimensional space. Allowance for correlation in time leads to adequate estimates of the conflict probability. As a result, we have obtained the stochastic differential equation for estimating the conflict probability (collision probability) and the equations for estimating the integral conflict probability and mean time to the first conflict.

Further studies are oriented toward numerical solution of the problem of estimating the characteristics of potential conflicts and statistical simulation of various situations. The practical solution of the problem is much simplified if the conflict is considered for the flight of two aircraft at the same altitude. One may even more reduce the order of the system of equations by accounting for the difference of control in the longitudinal and side directions of flight, namely, the factor of stabilization of a side position of the aircraft, and based on it using the random Ornstein–Uhlenbeck process to describe not the deviation velocity but the deviation itself.

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