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RESELLING OF OPTIONS AND CONVERGENCE OF OPTION REWARDS[1]

We consider the problem of optimal reselling of European options. A bivariate exponential diffusion process is used to describe the reselling model. In this way, the reselling problem is imbedded to the model of finding optimal reward for American type option based on this process. Convergence results are obtained for optimal reward functionals of American type options for perturbed multi-variate Markov processes. An approximation bivariate tree model is constructed and convergence of optimal expected reward for this tree model to the optimal expected reward for the corresponding American type option is proved.

1. Introduction

European options can only be exercised at maturity T, however there exists the possibility for the holder to sell the option on the second hand market. The question then arises at which moment of time is it optimal for the holder to sell the option, this is the reselling problem.

We use the classical geometric Brownian motion to model the price process and an exponential mean reverting Ornstein-Uhlenbeck process correlated with the price process to describe stochastic dynamics for implied volatility. We also assume that a market price for option is given by the Black–Scholes formula where implied volatility is used instead its initial value.

The problem of optimal reselling of European option is treated as the problem of finding optimal expected reward for American type option for this bivariate exponential diffusion process with asset price and implied volatility components.

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The paper contains five sections. In Section 2 we give exact formulation of reselling problem. In Section 3 we present general convergence results for reward functional of American type options for perturbed multivariate exponential Markov price processes. In Section 4 we specify these results for perturbed multivariate exponential price processes with independent increments. In Section 5 we construct the bivariate binomial-trinomial model approximating the bivariate diffusion process used to describe the reselling model and prove convergence of the optimal expected rewards for this tree model and the corresponding bivariate diffusion process.

It should be mentioned that our reselling model differs from those considered by Kukush, Mishura and Shevchenko (2006) where the usual geometric Brownian motion was used as a model for implied volatility. This model is too simple since it admit unbounded deviation of implied and initial volatilities. The reverting model for implied volatility introduced in the present paper is more complex but at the same time much more realistic.

Some general facts about expected reward functions in reselling model were derived in the paper mentioned above.

The present paper intends to solve another problem that is to build up an effective approximation algorithm for evaluation of optimal reward functionals for the reselling model.

We would like also to mention the papers by Silvestrov, Jönsson and Stenberg (2006, 2007, 2008) on optimal reward functionals for American type options for Markov price processes modulated by stochastic indices. Our general convergence results generalize the results of this paper to the multivariate Markov price processes.

In conclusion, we also would like to mention some papers by Amin and Khanna (1994), Silvestrov, Galochkin, and Sibirtsev (1999), Silvestrov, Galochkin, and Malyarenko (2000), Kukush and Silvestrov (2000, 2001, 2004), Jönsson (2001, 2005), Prigent (2003), Jönsson, Kukush, and Silvestrov (2004, 2005), Coquet and Toldo (2007) and Dupuis and Wang (2005) on convergence of American option rewards.

Optimal stopping problems for American type options have been also studied by Kim (1990), Jacka (1991), Peskir and Shiryaev (2006), by Zhang and Lim (2006) for models with stochastic volatility, by Gau, Huang and Subrahmanyam (2000) for American barrier options, by Lundgren (2007) for generalized American knock out option, by Shepp and Shiryaev (1993) for Russian options, and by Xia and Zhou (2007) for related stock loans models.

2. Formulation of reselling problem

We consider the classical geometric Brownian motion as price process given by the stochastic differential equation

$$d\ln S(t) = \mu dt + \sigma dW_1(t),\ t \geq 0, \tag{1}$$

where $\mu \in \mathbb{R}, \sigma > 0$; $W_1(t)$ is a standard Brownian motion, and the initial state $S(0) > 0$ is a constant. It is also assumed that the continuously compounded interest model with a riskless interest rate $r > 0$ is used.

In this case, the price at moment t for a European option, with the strike price $K > 0$ and maturity $T > 0$, is given by the celebrated Black–Scholes formula,

$$C(t, S(t), \sigma) = S(t)\Phi(d) - Ke^{-(T-t)}\Phi(d - \sigma\sqrt{T-t}), \tag{2}$$

where

$$d = \frac{\ln(S(t)/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}, \ \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

It is well known that the market price of European option deviates from the theoretical price. One of the explanations is that an implied volatility $\sigma(t)$ is used in formula (2) instead of σ.

We use a model given by the mean reverting Ornstein-Uhlenbeck process for implied volatility,

$$d(\ln\sigma(t) - \ln\sigma) = -\alpha(\ln\sigma(t) - \ln\sigma)dt + \nu dW_2(t), \ t \geq 0, \tag{3}$$

where $\alpha, \nu > 0$, $W_2(t)$ is also a standard Brownian motion, and the boundary condition $\sigma(0) = \sigma$.

Finally, we assume that the process $\vec{W}(t) = (W_1(t), W_2(t))$ is the bivariate Brownian motion with correlated components, i.e.,

$$\mathsf{E}W_1(t)W_2(t) = \rho t, \ t \geq 0, \tag{4}$$

where $\rho \in [-1, 1]$. It is useful to note that usually $\rho > 0$.

Note that the process $(S(t), \sigma(t))$ is a diffusion process.

The use of the market price $C(t, S(t), \sigma(t))$ actualizes the problem for reselling of European option. In this case it is assumed that an owner of the option can resell the option at some stopping time from the class $\mathcal{M}_T$ which includes all stopping times $0 \leq \tau \leq T$ that are Markov moments with respect to the filtration $\mathcal{F}_t = \sigma((S(s), \sigma(s)), s \leq t)$ generated by the vector process $(S(t), \sigma(t))$.

The object of our studies is the reward functional

$$\Phi(\mathcal{M}_T) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}e^{-r\tau}C(\tau, S(\tau), \sigma(\tau)). \tag{5}$$

Thus, the problem of reselling the European option is imbedded in the problem of optimal execution of American type option with the pay-off function $e^{-rt}C(t, S, \sigma)$ for the two-dimensional process $(S(t), \sigma(t))$.

Our approach is based on the approximation of process $(S(t), \sigma(t))$ by a properly fitted bivariate binomial-trinomial model. This approach require to solve three problems.

First, appropriate results concerning convergence for reward functional of American type options should be developed for multivariate Markov price processes.

Second, the bivariate binomial-trinomial model satisfying the corresponding recombination conditions and a polynomial (quadratic for bivariate trees) rate of growth of the number of nodes as function of the number of steps should be constructed.

Third, the conditions of convergence for reward functionals mentioned above should be verified.

We present the corresponding convergence results in the third and fourth section and the approximation results in the fifth section.

3. Convergence of rewards for multivariate Markov price processes

For every $\varepsilon \geq 0$, let $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), \dots, Y_k^{(\varepsilon)}(t))$, $t \geq 0$ be a càdlàg Markov process with the phase space $\mathbb{R}^k$ and transition probabilities $P^{(\varepsilon)}(t, \vec{y}, t+s, A)$. We interpret $\vec{Y}^{(\varepsilon)}(t)$ as a vector log-price process.

Now, we define a vector price process $\vec{S}^{(\varepsilon)}(t) = (S_1^{(\varepsilon)}(t), \dots, S_k^{(\varepsilon)}(t))$, $t \geq 0$ with the phase space $\mathbb{R}_+^k = \mathbb{R}_+ \times \dots \times \mathbb{R}_+$, where $\mathbb{R}_+ = (0, \infty)$, by the relations

$$S_i^{(\varepsilon)}(t) = e^{Y_i^{(\varepsilon)}(t)}, i = 1, \dots, k, \ t \geq 0. \tag{6}$$

Due to the one-to-one mapping and continuity property of exponential function, $\vec{S}^{(\varepsilon)}(t)$ is also a càdlàg Markov process.

For every $\varepsilon \geq 0$, let $g^{(\varepsilon)}(t, \vec{s})$, $(t, \vec{s}) \in \mathbb{R}_+ \times \mathbb{R}_+^k$ be a pay-off function. We assume that $g^{(\varepsilon)}(t, \vec{s})$ is a real valued Borel measurable function. Note that we do not assume pay-off functions to be non-negative.

Let $\mathcal{F}_t^{(\varepsilon)} = \sigma(\vec{Y}^{(\varepsilon)}(s), s \leq t)$ be the natural filtration of σ-fields, associated with the vector log-price process $\vec{Y}^{(\varepsilon)}(t), t \geq 0$. It is useful to note that this filtration coincides with the natural filtration generated by the price process $\vec{S}^{(\varepsilon)}(t), t \geq 0$.

We consider Markov moments $\tau^{(\varepsilon)}$ with respect to the filtration $\mathcal{F}_t^{(\varepsilon)}, t \geq 0$. It means that $\tau^{(\varepsilon)}$ is a random variable which takes values in $[0, \infty]$ and with the property $\{\omega : \tau^{(\varepsilon)}(\omega) \leq t\} \in \mathcal{F}_t^{(\varepsilon)}, t \geq 0$.

Let $\mathcal{M}_{max,T}^{(\varepsilon)}$ be the class of all Markov moments $\tau^{(\varepsilon)} \leq T$, where $T > 0$, and consider a class of Markov moments $\mathcal{M}_T^{(\varepsilon)} \subseteq \mathcal{M}_{max,T}^{(\varepsilon)}$.

Below we impose conditions on price processes and pay-off functions which guarantee that, for all ε small enough,

$$\sup_{\tau^{(\varepsilon)} \in \mathcal{M}_{max,T}^{(\varepsilon)}} \mathsf{E}|g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})| < \infty. \tag{7}$$

The main object of our studies is the reward functional, that is, the maximal expected pay-off over different classes of Markov moments, $\mathcal{M}_T^{(\varepsilon)}$,

$$\Phi(\mathcal{M}_T^{(\varepsilon)}) = \sup_{\tau^{(\varepsilon)} \in \mathcal{M}_T^{(\varepsilon)}} \mathsf{E} g^{(\varepsilon)}(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})). \tag{8}$$

We are interested in conditions of convergence for reward functionals for different classes of stopping times. In particular, we formulate conditions implying the following convergence relation:

$$\Phi(\mathcal{M}_{max,T}^{(\varepsilon)}) \to \Phi(\mathcal{M}_{max,T}^{(0)}) \text{ as } \varepsilon \to 0. \tag{9}$$

The first condition assumes the absolute continuity of pay-off functions and imposes power type upper bounds on their partial derivatives:

$\mathbf{A_1}$: There exists $\varepsilon_0 > 0$ such that for every $0 \leq \varepsilon \leq \varepsilon_0$: **(a)** function $g^{(\varepsilon)}(t, \vec{s})$ is absolutely continuous in t with respect to the Lebesgue measure on $[0, T]$ for every fixed $\vec{s} \in \mathbb{R}_+^k$ and in $\vec{s}$ with respect to the Lebesgue measure on $\mathbb{R}_+^k$ for every fixed $t \in [0, T]$; **(b)** for every $\vec{s} \in \mathbb{R}_+^k$, the partial derivative $|\frac{\partial g^{(\varepsilon)}(t,\vec{s})}{\partial t}| \leq K_1 + K_2 \sum_{j=1}^k s_j^{\gamma_0}$ for almost all $t \in [0, T]$ with respect to the Lebesgue measure on $[0, T]$, where $0 \leq K_1, K_2 < \infty$ and $\gamma_0 \geq 0$; **(c)** for every $t \in [0, T]$, the partial derivative $|\frac{\partial g^{(\varepsilon)}(t,\vec{s})}{\partial s_m}| \leq K_3 + K_4 \sum_{j=1}^k s_j^{\gamma_m}$ for almost all $\vec{s} \in \mathbb{R}_+^k$ with respect to the Lebesgue measure on $\mathbb{R}_+^k$, where $0 \leq K_3, K_4 < \infty$ and $\gamma_1, \ldots, \gamma_k \geq 0$, $m = 1, \ldots, k$. **(d)** for every $t \in [0, T]$, the function $g^{(\varepsilon)}(t, \vec{0}) = \overline{\lim}_{\vec{s} \to \vec{0}} g^{(\varepsilon)}(t, \vec{s}) \leq K_5$, where $0 \leq K_5 < \infty$.

It is useful to note that condition $\mathbf{A_1}$ implies that the function $g^{(\varepsilon)}(t, \vec{s})$ is continuous in $(t, \vec{s}) \in [0, T] \times \mathbb{R}_+^k$.

Denote $e^{\vec{y}} = (e^{y_1}, \ldots, e^{y_k})$. Condition $\mathbf{A_1}$ can obviously be re-written in the equivalent form in terms of the function $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A_1'}$: There exists $\varepsilon_0 > 0$ such that for every $0 \leq \varepsilon \leq \varepsilon_0$: **(a)** function $g^{(\varepsilon)}(t, e^{\vec{y}})$ is absolutely continuous in t with respect to the Lebesgue measure on $[0, T]$ for every fixed $\vec{y} \in \mathbb{R}_+^k$ and in $\vec{y}$ with respect to the Lebesgue measure on $\mathbb{R}_+^k$ for every fixed $t \in [0, T]$; **(b)** for every $\vec{y} \in \mathbb{R}_+^k$, the partial derivative in t is bounded as $\left|\frac{\partial g^{(\varepsilon)}(t,e^{\vec{y}})}{\partial t}\right| \leq K_1 + K_2 \sum_{j=1}^k e^{\gamma_0 y_j}$ for almost all $t \in [0, T]$ with respect to the Lebesgue measure on $[0, T]$, where $0 \leq K_1, K_2 < \infty$ and $\gamma_0 \geq 0$; **(c)** for every $t \in [0, T]$, the m-th partial derivative $\left|\frac{\partial g^{(\varepsilon)}(t,e^{\vec{y}})}{\partial y_m}\right| \leq (K_3 + K_4 \sum_{j=1}^k e^{\gamma_m y_j}) e^{y_m}$ for almost all $\vec{y} \in \mathbb{R}_+^k$ with respect to the Lebesgue measure on $\mathbb{R}_+^k$, where $0 \leq K_3, K_4 < \infty$ and $\gamma_1, \ldots, \gamma_k \geq 0$, $m = 1, \ldots, k$. **(d)** for every $t \in [0, T]$, the function $g^{(\varepsilon)}(t, -\infty, \ldots, -\infty) = \overline{\lim}_{y_i \to -\infty, i=\overline{1,k}} g^{(\varepsilon)}(t, e^{\vec{y}}) \leq K_5$, where $0 \leq K_5 < \infty$.

We use notations $\mathsf{E}_{\vec{y},t}$ and $\mathsf{P}_{\vec{y},t}$ for expectation and probability calculated under condition that $\vec{Y}^{(\varepsilon)}(t) = \vec{y}$.

For $\beta, c, T > 0, i = 1, \ldots, k$, define the exponential moment modulus of compactness for the càdlàg process $Y_i^{(\varepsilon)}(t), t \geq 0$,

$$\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) = \sup_{0 \leq t \leq t+u \leq t+c \leq T} \sup_{\vec{y} \in \mathbb{R}_+^k} \mathsf{E}_{\vec{y},t}(e^{\beta|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)|} - 1).$$

We use the following conditions for exponential moment modulus of compactness for log-price processes:

$\mathbf{C_1}$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon \to 0} \sum_{i=1}^k \Delta_\beta(Y_i^{(\varepsilon)}(\cdot), c, T) = 0$ for some $\beta > \gamma = \max(\gamma_0, \gamma_1 + 1, \ldots, \gamma_k + 1)$, where γ_0 and $\gamma_1, \ldots, \gamma_k$ are the parameters introduced in condition $\mathbf{A_1}$,

and

$\mathbf{C_2}$: $\overline{\lim}_{\varepsilon \to 0} \mathsf{E} e^{\beta |Y_i^{(\varepsilon)}(0)|} < \infty$, $i = 1, \ldots, k$, where β is the parameter introduced in condition $\mathbf{C_1}$.

The following lemma gives asymptotically uniform upper bounds for moments of maximum of price processes, with respect to perturbation parameter.

Lemma 1. *Let conditions $\mathbf{A_1}$, $\mathbf{C_1}$, and $\mathbf{C_2}$ hold. Then there exists a constant $L_1 < \infty$ such that for every $\varepsilon \leq \varepsilon_1$,*

$$\sup_{\tau^{(\varepsilon)} \in \mathcal{M}_{max,T}^{(\varepsilon)}} \mathsf{E}|g(\tau^{(\varepsilon)}, \vec{S}^{(\varepsilon)}(\tau^{(\varepsilon)})| \leq \mathsf{E} \sup_{0 \leq u \leq T} |g^{(\varepsilon)}(u, \vec{S}^{(\varepsilon)}(u))|^{\frac{\beta}{\gamma}} \leq L_1. \quad (10)$$

Let $\Pi = \{0 = t_0 < t_1 < \ldots t_N = T\}$ be a partition of the interval $[0, T]$ and $d(\Pi) = \max_{1 \leq i \leq N}(t_i - t_{i-1})$. We consider the class $\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}$ of all Markov moments from $\mathcal{M}_{max,T}^{(\varepsilon)}$, which only take the values $t_0, t_1, \ldots t_N$, and the class $\mathcal{M}_{\Pi,T}^{(\varepsilon)}$ of all Markov moments $\tau^{(\varepsilon)}$ from $\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}$ such that the event $\{\omega : \tau^{(\varepsilon)}(\omega) = t_j\} \in \sigma(\vec{Y}^{(\varepsilon)}(t_0), \ldots, \vec{Y}^{(\varepsilon)}(t_j))$ for $j = 0, \ldots N$. By definition,

$$\mathcal{M}_{\Pi,T}^{(\varepsilon)} \subseteq \hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)} \subseteq \mathcal{M}_{max,T}^{(\varepsilon)}. \quad (11)$$

Relations (11) imply that, under conditions of Lemma 1,

$$-\infty < \Phi(\mathcal{M}_{\Pi,T}^{(\varepsilon)}) \leq \Phi(\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)}) \leq \Phi(\mathcal{M}_{max,T}^{(\varepsilon)}) < \infty. \quad (12)$$

The reward functionals $\Phi(\mathcal{M}_{max,T}^{(\varepsilon)})$, $\Phi(\hat{\mathcal{M}}_{\Pi,T}^{(\varepsilon)})$, and $\Phi(\mathcal{M}_{\Pi,T}^{(\varepsilon)})$ correspond to American type option in continuous time, Bermudan type option in continuous time, and American type option in discrete time, respectively.

In the first two cases, the underlying price process is a continuous time Markov type price process, while in the third case the corresponding price process is a discrete time Markov type process.

The random variables $\vec{Y}^{(\varepsilon)}(t_0), \vec{Y}^{(\varepsilon)}(t_1), \ldots, \vec{Y}^{(\varepsilon)}(t_N)$ are connected in a discrete time inhomogeneous Markov chain with the phase space $\mathbb{R}^k$, transition probabilities $P^{(\varepsilon)}(t_n, \vec{y}, t_{n+1}, A)$, and initial distribution $P^{(\varepsilon)}(A)$. Note that we have slightly modified the standard definition of a discrete time Markov chain by counting moments $t_0, \ldots, t_N$ as the moments of jumps for the Markov chain $Y^{(\varepsilon)}(t_n)$ instead of the moments $0, \ldots, N$. This is done in order to synchronize the discrete and continuous time models. Thus, the optimization problem (8) for the class $\mathcal{M}_{\Pi,T}$ is really a problem of optimal expected reward for American type options in discrete time.

The following lemma establishes useful equality between reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T})$ and $\Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T})$.

Lemma 2. *Let conditions of Lemma 1 hold. Then for any partition* $\Pi = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ *of interval* $[0, T]$,

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) = \Phi(\hat{\mathcal{M}}^{(\varepsilon)}_{\Pi,T}). \tag{13}$$

The following theorem gives a skeleton approximation for reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{max,T})$ which is asymptotically uniform with respect to perturbation parameter.

Theorem 1. *Let conditions* $\mathbf{A_1}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold, then there exist constants* $L_2, L_3 < \infty$ *such that the following skeleton approximation inequality holds, for every* $\varepsilon \leq \varepsilon_1$*:,*

$$\begin{aligned} &\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T}) \\ &\quad \leq L_2 d(\Pi) + L_3 \Big(\sum_{i=1}^{k} \Delta_\beta(Y^{(\varepsilon)}_i(\cdot), d(\Pi), T)\Big)^{\frac{\beta-\gamma}{\beta}}. \end{aligned} \tag{14}$$

Let us now formulate conditions of convergence for discrete time reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi,T})$ for a given partition $\Pi = \{0 = t_0 < t_1 \ldots < t_N = T\}$ of interval $[0, T]$. In this case it is natural to use conditions based on the transition probabilities between the sequential moments of this partition and values of the pay-off functions at the moments of this partition. Condition $\mathbf{A_1}$ is replaced by a simpler condition:

- $\mathbf{A_2}$: There exists $\varepsilon_2 > 0$ such that, for every $0 \leq \varepsilon \leq \varepsilon_2$, function $g^{(\varepsilon)}(t_n, \vec{s}) \leq K_5 + K_6 \sum_{i=1}^k s_i^\gamma$, for $n = 0, \ldots, N$ and $\vec{s} \in \mathbb{R}_{k+}$ for some $\gamma \geq 1$ and constants $K_5, K_6 < \infty$.

Obviously, condition $\mathbf{A_2}$ can be re-written in terms of functions $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A_2'}$: There exists $\varepsilon_0 > 0$ such that, for every $0 \leq \varepsilon \leq \varepsilon_0$, function $g^{(\varepsilon)}(t_n, e^{\vec{y}}) \leq K_5 + K_6 \sum_{i=1}^{k} e^{\gamma y_i}$, for $n = 0, \ldots, N$ and $\vec{y} \in \mathbb{R}_k$ for some $\gamma \geq 1$ and constants $K_5, K_6 < \infty$.

We also need an assumption about convergence of pay-off functions. We require local uniform convergence for pay-off functions on some sets, which later will be assumed to have the value 1 for the corresponding limit transition probabilities and the limit initial distribution:

$\mathbf{A_3}$: There exists a measurable set $\mathbb{S}'_{t_n} \subseteq \mathbb{R}_{k+}$ for every $n = 0, \ldots, N$, such that $g^{(\varepsilon)}(t_n, \vec{s}^{\,(\varepsilon)}) \to g^{(0)}(t_n, \vec{s})$ as $\varepsilon \to 0$, for any $\vec{s}^{\,(\varepsilon)} \to \vec{s} \in \mathbb{S}'_{t_n}$ and $n = 0, \ldots, N$.

Obviously, condition $\mathbf{A_3}$ can be re-written in terms of functions $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A_3'}$: There exists measurable sets $\mathbb{Y}'_{t_n} \subseteq \mathbb{R}^k$, $n = 0, \ldots, N$, such that $g^{(\varepsilon)}(t_n, e^{\vec{y}^{\,(\varepsilon)}}) \to g^{(0)}(t_n, e^{\vec{y}})$ as $\varepsilon \to 0$, for any $\vec{y}^{\,(\varepsilon)} \to \vec{y} \in \mathbb{Y}'_{t_n}$ and $n = 0, \ldots, N$.

Let us now formulate conditions assumed for the transition probabilities and initial distributions of the process $\vec{Y}^{(\varepsilon)}(t)$.

Symbol $\Rightarrow$ is used below to denote weak convergence of probability measures, i.e. convergence of their values at sets of continuity for the corresponding limit measure.

The first condition assumes weak convergence of the transition probabilities that should be locally uniform with respect to initial states from some sets, and also that the corresponding limit measures are concentrated on these sets:

$\mathbf{B_1}$: There exist measurable sets $\mathbb{Y}''_{t_n} \subseteq \mathbb{R}^k$, $n = 0, \ldots, N$ such that: **(a)** $P^{(\varepsilon)}(t_n, \vec{y}^{\,(\varepsilon)}, t_{n+1}, \cdot) \Rightarrow P^{(0)}(t_n, \vec{y}, t_{n+1}, \cdot)$ as $\varepsilon \to 0$, for any $\vec{y}^{\,(\varepsilon)} \to \vec{y} \in \mathbb{Y}_{t_n}$ as $\varepsilon \to 0$ and $n = 0, \ldots, N-1$; **(b)** $P^{(0)}(t_n, \vec{y}, t_{n+1}, \mathbb{Y}'_{t_{n+1}} \cap \mathbb{Y}''_{t_{n+1}}) = 1$ for every $\vec{y} \in \mathbb{Y}''_{t_n}$ and $n = 0, \ldots, N-1$.

The second condition assumes weak convergence of the initial distributions to some distribution that is assumed to be concentrated on the sets of convergence for the corresponding transition probabilities:

$\mathbf{B_2}$: **(a)** $P^{(\varepsilon)}(\cdot) \Rightarrow P^{(0)}(\cdot)$ as $\varepsilon \to 0$; **(b)** $P^{(0)}(\mathbb{Y}_0) = 1$, where $\mathbb{Y}_0$ is the set introduced in condition $\mathbf{B_1}$.

We also weaken condition $\mathbf{C_1}$ by replacing it by a simpler condition, which is implied by condition $\mathbf{C_1}$:

$\mathbf{C_3}$: $\overline{\lim}_{\varepsilon \to 0} \sup_{\vec{y} \in \mathbb{R}^k} \mathsf{E}_{\vec{y}, t_n} \sum_{i=1}^{k} (e^{\beta |Y_i^{(\varepsilon)}(t_{n+1}) - Y_i^{(\varepsilon)}(t_n)|} - 1) < \infty$, $n = 0, \ldots, N-1$, for some $\beta > \gamma$, where γ is the parameter introduced in condition $\mathbf{A_2}$.

Condition $\mathbf{C_2}$ does not change and takes the following form:

$\mathbf{C_4}$: $\overline{\lim}_{\varepsilon\to 0} \mathsf{E} \sum_{i=1}^{k} e^{\beta |Y_i^{(\varepsilon)}(t_0)|} < \infty$, where β is the parameter introduced in condition $\mathbf{C_3}$.

The following theorem gives conditions of convergence for reward functionals $\Phi(\mathcal{M}_{\Pi,T}^{(\varepsilon)})$ for a given partition Π.

Theorem 2. *Let conditions* $\mathbf{A_2}$*,* $\mathbf{A_3}$*,* $\mathbf{B_1}$*,* $\mathbf{B_2}$*,* $\mathbf{C_3}$*, and* $\mathbf{C_4}$ *hold. Then, the following asymptotic relation holds for the partition* $\Pi = \{0 = t_0 < t_1 \cdots < t_N = T\}$ *of interval* $[0, T]$*,*

$$\Phi(\mathcal{M}_{\Pi,T}^{(\varepsilon)}) \to \Phi(\mathcal{M}_{\Pi,T}^{(0)}) \text{ as } \varepsilon \to 0. \tag{15}$$

Let now formulate conditions of convergence for discrete time reward functionals $\Phi(\mathcal{M}_{max,T}^{(\varepsilon)})$ for continuous time model.

As was mentioned above, in the discrete time case, the pay-off functions can be discontinuous. In the continuous time case, the derivatives of the pay-off functions are involved in condition $\mathbf{A_1}$. The corresponding assumptions imply continuity of the pay-off functions.

This give us possibility to weaken the assumption concerning the convergence of the pay-off functions and just to require their pointwise convergence:

$\mathbf{A_4}$: $g^{(\varepsilon)}(t, \vec{s}) \to g^{(0)}(t, \vec{s})$ as $\varepsilon \to 0$, for every $(t, \vec{s}) \in [0, T] \times \mathbb{R}_{k+}$.

Obviously, condition $\mathbf{A_4}$ can be re-written in terms of function $g^{(\varepsilon)}(t, e^{\vec{y}})$:

$\mathbf{A}'_4$: $g^{(\varepsilon)}(t, e^{\vec{y}}) \to g^{(0)}(t, e^{\vec{y}})$ as $\varepsilon \to 0$, for every $(t, \vec{y}) \in [0, T] \times \mathbb{R}^k$.

Let us now formulate conditions assumed for the transition probabilities and the initial distributions of process $\vec{Y}^{(\varepsilon)}(t)$.

The first condition assumes weak convergence of the transition probabilities that should be locally uniform with respect to initial states from some sets, and also that the corresponding limit measures are concentrated on these sets:

$\mathbf{B_3}$: There exist measurable sets $\mathbb{Y}_t \subseteq \mathbb{R}^k$, $t \in [0, T]$ such that: **(a)** $P^{(\varepsilon)}(t, \vec{y}^{\,(\varepsilon)}, t+u, \cdot) \Rightarrow P^{(0)}(t, \vec{y}, t+u, \cdot)$ as $\varepsilon \to 0$, for any $\vec{y}^{\,(\varepsilon)} \to \vec{y} \in \mathbb{Y}_t$ as $\varepsilon \to 0$ and $0 \leq t < t + u \leq T$; **(b)** $P^{(0)}(t, \vec{y}, t + u, \mathbb{Y}_{t+u}) = 1$ for every $\vec{y} \in \mathbb{Y}_t$ and $0 \leq t < t + u \leq T$.

The second condition assumes weak convergence of the initial distributions to some distribution that is assumed to be concentrated on the sets of convergence for the corresponding transition probabilities:

$\mathbf{B_4}$: **(a)** $P^{(\varepsilon)}(\cdot) \Rightarrow P^{(0)}(\cdot)$ as $\varepsilon \to 0$; **(b)** $P^{(0)}(\mathbb{Y}_0) = 1$, where $\mathbb{Y}_0$ is the set introduced in condition $\mathbf{B_3}$.

The following theorem presents our main convergence result. It gives conditions of convergence for reward functionals $\Phi(\mathcal{M}^{(\varepsilon)}_{max,T})$.

Theorem 3. *Let conditions* $\mathbf{A_1}$, $\mathbf{A_4}$, $\mathbf{B_3}$, $\mathbf{B_4}$, $\mathbf{C_1}$, *and* $\mathbf{C_2}$ *hold. Then*

$$\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) \to \Phi(\mathcal{M}^{(0)}_{max,T}) < \infty \text{ as } \varepsilon \to 0. \tag{16}$$

Here we just give a sketch of the proof that show in which way Theorems 1 and 2 are used in the proof of Theorem 3.

Choose some sequence of partitions Π_N such that $d(\Pi_N) \to 0$ as $N \to \infty$.

Theorem 1 implies that, under conditions $\mathbf{A_1}$, $\mathbf{C_1}$, and $\mathbf{C_2}$, the following relation holds:

$$\lim_{N\to\infty} \overline{\lim_{\varepsilon\to 0}} |\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T})| = 0. \tag{17}$$

It is also not difficult to check that conditions of Theorem 3 imply that conditions of Theorem 2 hold for any partition Π of the interval $[0, T]$, and thus, according to Theorem 2, for any $N = 1, 2, \ldots$,

$$\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T}) \to \Phi(\mathcal{M}^{(0)}_{\Pi_N,T}) \text{ as } \varepsilon \to 0. \tag{18}$$

The following inequality can be written down for any partition Π_N:

$$\begin{aligned} |\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{max,T})| &\leq |\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T})| \\ &+ |\Phi(\mathcal{M}^{(0)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{\Pi_N,T})| + |\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T}) - \Phi(\mathcal{M}^{(0)}_{\Pi_N,T})|. \end{aligned} \tag{19}$$

Using this inequality and relations (17) and (18) we get the following asymptotic bound:

$$\begin{aligned} &\overline{\lim_{\varepsilon\to 0}} |\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{max,T})| \\ &\quad \leq \overline{\lim_{\varepsilon\to 0}} |\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_N,T})| + |\Phi(\mathcal{M}^{(0)}_{max,T}) - \Phi(\mathcal{M}^{(0)}_{\Pi_N,T})|. \end{aligned} \tag{20}$$

Finally, relation (17) implies (note that relation $\varepsilon \to 0$ admits also the case where $\varepsilon = 0$) that the expression on the right hand side in (20) can be forced to take a value less then any $\delta > 0$ by choosing the partition Π_N with the diameter $d(\Pi_N)$ small enough. This proves the asymptotic relation (16) and completes the proof.

In conclusion of this section, let us formulate some useful sufficient conditions for an important condition of moment compactness $\mathbf{C_1}$.

Let us introduce the modulus of J-compactness, for $h, c > 0, i = 1, \ldots, k$,

$$\Delta(Y^{(\varepsilon)}_i(\cdot), h, c, T) = \sup_{0\leq t\leq t+u\leq t+c\leq T} \sup_{\vec{y}\in\mathbb{R}^k} \mathsf{P}_{\vec{y},t}\{|Y^{(\varepsilon)}_i(t+u) - Y^{(\varepsilon)}_i(t)| \geq h\}.$$

The following condition of J-compactness plays the key role in functional limit theorems for Markov type càdlàg processes:

C$_5$: $\lim_{c\to 0} \overline{\lim}_{\varepsilon\to 0} \Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T) = 0,\ h > 0, i = 1, \dots, k$.

Introduce also the quantity, which represents the maximum of moment generating functions for increments of the log-price processes $Y_i^{(\varepsilon)}(t)$, $i = 1, \dots, k$,

$$\Xi_\beta(Y_i^{(\varepsilon)}(\cdot), T) = \sup_{0\le t\le t+u\le T} \sup_{\vec{y}\in\mathbb{R}^k} \mathsf{E}_{\vec{y},t} e^{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))},\ \beta \in \mathbb{R}_1.$$

The following condition formulated in terms of these moment generating functions can be effectively verified in many cases:

C$_6$: $\overline{\lim}_{\varepsilon\to 0} \Xi_{\pm\beta'}(Y_i^{(\varepsilon)}(\cdot), T) < \infty, i = 1, \dots, k$, for some $\beta' > \beta$, where β is the parameter penetrating condition **C$_1$**.

Lemma 3. *Conditions* **C$_5$** *and* **C$_6$** *imply condition* **C$_1$**.

3. Exponential price processes with independent increments

In this section, examples illustrate the results given in Theorems 1–3.

Let us consider the model where the log-price process $\vec{Y}^{(\varepsilon)}(t), t \ge 0$ is a càdlàg processes with independent increments. We also assume for simplicity that the initial state of process $\vec{Y}^{(\varepsilon)}(0) = \vec{y}^{(\varepsilon)} = (y_i^{(\varepsilon)}, i = 1, \dots, k)$ is a constant. The process $\vec{Y}^{(\varepsilon)}(t)$ is a càdlàg Markov process with transition probabilities which are connected with the distributions of increments for this process $P^{(\varepsilon)}(t, t+u, A)$ by the following relation,

$$P^{(\varepsilon)}(t, \vec{y}, t+u, A) = P^{(\varepsilon)}(t, t+u, A - \vec{y}) = \mathsf{P}\{\vec{y} + \vec{Y}^{(\varepsilon)}(t+u) - \vec{Y}^{(\varepsilon)}(t) \in A\}. \quad (21)$$

Let us assume the following standard condition of weak convergence for distributions of increments for log-price processes:

D$_1$: $P^{(\varepsilon)}(t, t+u, \cdot) \Rightarrow P^{(0)}(t, t+u, \cdot)$ as $\varepsilon \to 0$, $0 \le t \le t+u \le T$.

Representation (21) implies in an obvious way that condition **B$_3$** holds with the sets $\mathbb{Y}_t = \mathbb{R}^k, t \in [0, T]$, i.e., distributions of increments for the processes $Y_i^{(\varepsilon)}(t)$ locally uniformly weakly converge if and only if condition **D$_1$** holds. Thus, in the case of processes with independent increments, the condition **B$_3$** with the sets $\mathbb{Y}_t = \mathbb{R}^k$ pointed above is, in fact, equivalent to the standard condition of weak convergence for such processes.

In this case the J-compactness modulus $\Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T)$ takes the following form:

$$\Delta(Y_i^{(\varepsilon)}(\cdot), h, c, T) = \sup_{0\le t\le t+u\le t+c\le T} \mathsf{P}\{|Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t)| \ge h\}.$$

Thus, condition **C$_5$** is reduced to the standard J-compactness condition for the log-price processes:

D₂: $\overline{\lim}_{c\to 0}\lim_{\varepsilon\to 0}\Delta(Y_i^{(\varepsilon)}(t),h,c,T)=0,\ h>0, i=1,\ldots,k.$

Note that conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ imply J-convergence of processes $\vec{Y}^{(\varepsilon)}(t)$, $t\in[0,T]$ to process $\vec{Y}^{(0)}(t), t\in[0,T]$ as $\varepsilon\to 0$ and stochastic continuity of the limit process.

Also, the quantities $\Xi_\beta(Y_i^{(\varepsilon)}(\cdot),T), i=1,\ldots,k$ take a simplified form,

$$\Xi_\beta(Y_i^{(\varepsilon)}(\cdot),T)=\sup_{0\le t\le t+u\le T}\mathsf{E}e^{\beta(Y_i^{(\varepsilon)}(t+u)-Y_i^{(\varepsilon)}(t))},\ \beta\in\mathbb{R}_1.$$

Therefore, condition $\mathbf{C_6}$ takes the following form:

$\mathbf{D_3}$: $\overline{\lim}_{\varepsilon\to 0}\Xi_{\pm\beta'}(Y_i^{(\varepsilon)}(\cdot),T)<\infty, i=1,\ldots,k$, for some $\beta'>\beta$, where β is the parameter penetrating condition $\mathbf{C_1}$.

According to Lemma 3, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ imply condition $\mathbf{C_1}$.

Condition $\mathbf{B_4}$ is reduced in this case to the following condition:

$\mathbf{D_4}$: $\lim_{\varepsilon\to 0}\vec{y}^{(\varepsilon)}=\vec{y}_0$.

Note that $\vec{y}_0$ can be any real vector since the set $\mathbb{Y}_0=\mathbf{R}_k$.

Obviously, condition $\mathbf{D_4}$ implies also condition $\mathbf{C_4}$.

Summarizing the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 3 hold for the exponential price processes with independent increments $\vec{S}^{(\varepsilon)}(t), t\in[0,T]$, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{D_1}$ – $\mathbf{D_4}$ hold.

The skeleton approximations $\vec{Y}^{(\varepsilon)}(t)=\vec{Y}^{(0)}([t/\varepsilon]),\ t\ge 0$ for a stochastically continuous càdlàg log-price process $\vec{Y}^{(0)}(t),\ t\ge 0$ with independent increments give an example of the model introduced above.

In this case, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ automatically hold.

As far as condition $\mathbf{D_3}$ is concerned it is implied by the following condition:

$\mathbf{D_5}$: $\Xi_{\pm\beta'}(Y_i^{(0)}(\cdot),T)<\infty, i=1,\ldots,k$, for some $\beta'>\beta$, where β is the parameter penetrating condition $\mathbf{C_1}$.

Thus, if we let conditions $\mathbf{A_1}$, $\mathbf{A_4}$, and $\mathbf{D_5}$ hold, then the statement of Theorem 3 holds for the exponential price processes $\vec{S}^{(\varepsilon)}(t)=e^{\vec{Y}^{(0)}([t/\varepsilon])}, t\in[0,T]$.

It is not out of picture to note that, in this case, Theorem 1 yields a stronger result in the form of explicit estimates for the accuracy of skeleton approximations for reward functions.

Note also that the optimal expected rewards for the skeleton price processes $\vec{S}^{(\varepsilon)}(t)=e^{\vec{Y}^{(0)}([t/\varepsilon])}$ can be estimated with the use of Monte Carlo simulation.

We refer to the papers by Boyle, Broadie and Glasserman (1997), Jönsson (2001, 2005), Jönsson, Kukush and Silvestrov (2004, 2005), Longstaff and

Schwartz (2001) , Lundgren (2007), Silvestrov, Galochkin and Malyarenko (2001) and Silvestrov, Galochkin and Sibirtsev (1999), where such Monte Carlo based algorithms are discussed.

Combination of these Monte Carlo based estimates with the skeleton approximations described above yields the effective approximation methods for optimal expected rewards for American type options with non-standard pay-offs.

In order to illustrate the results presented above, let us consider the a bivariate geometric Brownian price process $\vec{S}^{(0)}(t) = e^{\vec{Y}^{(0)}(t)}, t \geq 0$, where $\vec{Y}^{(0)}(t) = (Y_1^{(0)}(t), Y_2^{(0)}(t)), t \geq 0$ is a bivariate Brownian motion with components

$$Y_i^{(0)}(t) = y_i^{(0)} + \mu_i t + \sigma_i W_i(t),\ t \geq 0,\ i = 1, 2, \tag{22}$$

which are correlated, i.e., $\mathsf{E} W_1(t) W_2(t) = \rho t, t \geq 0$.

We approximate the process $\vec{Y}^{(0)}(t), t \geq 0$ with a bivariate binomial sum-process $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), Y_2^{(\varepsilon)}(t)), t \geq 0$ with components

$$Y_i^{(\varepsilon)}(t) = y_i^{(0)} + \sum_{1 \leq k \leq [t/\varepsilon]} Y_{n,i}^{(\varepsilon)},\ t \geq 0,\ i = 1, 2. \tag{23}$$

Here $\varepsilon > 0$ and $\vec{Y}_n^{(\varepsilon)} = (Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$, $n = 1, 2, \ldots$ are i.i.d. random vectors which take values $(+u_1^{(\varepsilon)}, +u_2^{(\varepsilon)})$, $(+u_1^{(\varepsilon)}, -u_2^{(\varepsilon)})$, $(-u_1^{(\varepsilon)}, +u_2^{(\varepsilon)})$, and $(-u_1^{(\varepsilon)}, -u_2^{(\varepsilon)})$ with probabilities $p_{++}^{(\varepsilon)}$, $p_{+-}^{(\varepsilon)}$, $p_{-+}^{(\varepsilon)}$, and $p_{--}^{(\varepsilon)}$, respectively.

Note that we assumed for simplicity that the initial state $\vec{Y}^{(\varepsilon)}(0) = \vec{y}^{(0)} = (y_1^{(0)}, y_2^{(0)})$ is a constant which does not depend of ε. This automatically implies that conditions $\mathbf{B_4}$ and $\mathbf{C_4}$ hold.

In order to fit the bivariate binomial sum-processes defined above to the limit bivariate Brownian motion, we should fit expectations, variances, and covariance coefficients for summands $(Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$ (whose distributions do not depend on n) to the corresponding quantities for the increments of the bivariate Brownian motion $(\mu_1 \varepsilon + \sigma_1(W_1((i+1)\varepsilon) - W_1(i\varepsilon)), \mu_2 \varepsilon + \sigma_2(W_2((i+1)\varepsilon) - W_2(i\varepsilon)))$ (whose distributions do not depend on i).

The following system of six equations with six unknowns should be solved:

$$\begin{cases} \mathsf{E}[Y_{1,1}^{(\varepsilon)}] & = u_1^{(\varepsilon)}(2(p_{++}^{(\varepsilon)} + p_{+-}^{(\varepsilon)}) - 1) & = \mu_1 \varepsilon, \\ \mathsf{V}ar[Y_{1,1}^{(\varepsilon)}] & = (u_1^{(\varepsilon)})^2 - (\mu_1 \varepsilon)^2 & = \sigma_1^2 \varepsilon, \\ \mathsf{E}[Y_{1,2}^{(\varepsilon)}] & = u_2^{(\varepsilon)}(2(p_{++}^{(\varepsilon)} + p_{-+}^{(\varepsilon)}) - 1) & = \mu_2 \varepsilon, \\ \mathsf{V}ar[Y_{1,2}^{(\varepsilon)}] & = (u_2^{(\varepsilon)})^2 - (\mu_2 \varepsilon)^2 & = \sigma_2^2 \varepsilon, \\ \mathsf{C}ov[Y_{1,1}^{(\varepsilon)}, Y_{1,2}^{(\varepsilon)}] & = \frac{u_1^{(\varepsilon)} u_2^{(\varepsilon)} (p_{++}^{(\varepsilon)} + p_{--}^{(\varepsilon)} - p_{-+}^{(\varepsilon)} - p_{+-}^{(\varepsilon)}) - \mu_1 \mu_2 \varepsilon^2}{\sigma_1 \sigma_2 \varepsilon} & = \rho, \\ & p_{++}^{(\varepsilon)} + p_{-+}^{(\varepsilon)} + p_{-+}^{(\varepsilon)} + p_{--}^{(\varepsilon)} & = 1. \end{cases} \tag{24}$$

This system has, for every $\varepsilon > 0$, the following unique solution,

$$
\begin{cases}
u_1^{(\varepsilon)} & = \sqrt{\varepsilon}\sqrt{\sigma_1^2 + \mu_1^2\varepsilon}, \\
u_2^{(\varepsilon)} & = \sqrt{\varepsilon}\sqrt{\sigma_2^2 + \mu_2^2\varepsilon}, \\
p_{++}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(\frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} + \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) + \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}, \\
p_{+-}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(\frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} - \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) - \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}, \\
p_{-+}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(-\frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} + \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) - \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}, \\
p_{--}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\sqrt{\varepsilon}\Big(-\frac{\mu_1}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}} - \frac{\mu_2}{\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}\Big) + \frac{1}{4}\frac{\rho\sigma_1\sigma_2+\mu_1\mu_2\varepsilon}{\sqrt{\sigma_1^2+\mu_1^2\varepsilon}\sqrt{\sigma_2^2+\mu_2^2\varepsilon}}.
\end{cases} \tag{25}
$$

It is useful to note that the corresponding parameter have the following asymptotic expansions,

$$
\begin{cases}
u_1^{(\varepsilon)} & = \sqrt{\varepsilon}\sigma_1 + o(\varepsilon), \\
u_2^{(\varepsilon)} & = \sqrt{\varepsilon}\sigma_2 + o(\varepsilon), \\
p_{++}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\rho + \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} + \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon), \\
p_{+-}^{(\varepsilon)} & = \frac{1}{4} - \frac{1}{4}\rho + \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} - \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon), \\
p_{-+}^{(\varepsilon)} & = \frac{1}{4} - \frac{1}{4}\rho - \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} - \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} - \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon), \\
p_{--}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\rho - \frac{1}{4}\Big(\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2}\Big)\sqrt{\varepsilon} + \frac{1}{4}\Big(\frac{\mu_1\mu_2}{\sigma_1\sigma_2} + \frac{\rho\mu_1^2}{2\sigma_1^2} - \frac{\rho\mu_2^2}{2\sigma_2^2}\Big)\varepsilon + o(\varepsilon).
\end{cases} \tag{26}
$$

Relation (26) guarantees that probabilities in (25) take values in interval $(0, 1)$ for ε small enough in the most interesting non-degenerate case, where $|\rho| < 1$. Note also that the sum of these probabilities is equal to 1, according to the last equation in (24).

The problem can be however reduced to more simple case where drift coefficients $\mu_1 = \mu_2 = 0$ and the initial state $\vec{y}^{(0)} = (0, 0)$.

Let us consider processes $\tilde{\vec{S}}^{(0)}(t) = e^{\tilde{\vec{Y}}^{(0)}(t)}, t \geq 0$, where the log-price process $\tilde{\vec{Y}}^{(0)}(t) = (\tilde{Y}_1^{(0)}(t), \tilde{Y}_2^{(0)}(t)), t \geq 0$ is a bivariate Brownian motion with components $\tilde{Y}_i^{(0)}(t) = \sigma_i W_i(t), t \geq 0, i = 1, 2$, which are correlated, i.e., $\mathsf{E}W_1(t)W_2(t) = \rho t, t \geq 0$. Obviously, the natural filtration $\mathcal{F}_t, t \geq 0$ is the same for processes $\vec{S}^{(0)}(t), t \geq 0$ and $\tilde{\vec{S}}^{(0)}(t), t \geq 0$.

Let $g^{(\varepsilon)}(t, \vec{s}) = g^{(\varepsilon)}(t, (s_1, s_2))$ be payoff functions that satisfy conditions $\mathbf{A_1}$ and $\mathbf{A_4}$. Let us now consider the transformed payoff functions $\tilde{g}^{(\varepsilon)}(t, \vec{s}) = g^{(\varepsilon)}(t, (e^{y_1^{(0)} + \mu_1 t}s_1, e^{y_2^{(0)} + \mu_2 t}s_2))$. These functions also satisfy conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ with some constants $K_i, i = 1, \ldots, 5$ and parameters $\gamma_0 = \gamma = \max(\gamma_0, \gamma_1 + 1, \gamma_2 + 1)$ and the same parameters γ_1 and γ_2.

It readily follows from the remarks above that the reward functional

$$
\Phi(\mathcal{M}_T^{(0)}) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}g^{(0)}(\tau, \vec{S}^{(0)}(\tau)) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}\tilde{g}^{(0)}(\tau, \tilde{\vec{S}}^{(0)}(\tau)). \tag{27}
$$

Now, we can approximate the bivariate Brownian processes $\vec{\tilde{Y}}^{(0)}(t)$ by the corresponding bivariate sum-processes $\vec{\tilde{Y}}^{(\varepsilon)}(t)$ as it was described above. In this case however the parameters μ_1 and μ_2 will take the value 0 in systems of equations (25) and (26). In this case, the solution to these systems will take the following simpler form,

$$\begin{cases} u_1^{(\varepsilon)} & = \sqrt{\varepsilon}\sigma_1, \\ u_2^{(\varepsilon)} & = \sqrt{\varepsilon}\sigma_2, \\ p_{++}^{(\varepsilon)} = p_{--}^{(\varepsilon)} & = \frac{1}{4} + \frac{1}{4}\rho, \\ p_{+-}^{(\varepsilon)} = p_{-+}^{(\varepsilon)} & = \frac{1}{4} - \frac{1}{4}\rho. \end{cases} \tag{28}$$

The probabilities in (28) take non-negative values for any $|\rho| \leq 1$.

By applying convergence theorems for vector sum-processes with independent increments given in Skorokhod (1964) it is easy to check that the processes $\vec{Y}^{(\varepsilon)}(t), t \in [0, T]$ with parameters given in (25) weakly and J-converge to process $\vec{Y}^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$.

These statements remain true also if parameters of the approximating processes would be chosen equal to the corresponding sums of terms in the asymptotic expansions (26) with omitted terms $o(\varepsilon)$.

Also the processes $\vec{\tilde{Y}}^{(\varepsilon)}(t), t \in [0, T]$ with parameters given in (28) weakly and J-converge to process $\vec{\tilde{Y}}^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$.

Thus, conditions $\mathbf{D_1}$ and $\mathbf{D_2}$ hold for processes $\vec{Y}^{(\varepsilon)}(t)$.

Also, the moment generation function $\mathsf{E}\exp\{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))\}$ exists for any $\beta \in \mathbb{R}$ and has an explicit form. For example, in the case of pre-limit processes with parameters defined in (25) and the corresponding limit processes, they have the following form, for $0 \leq t \leq t+u \leq T$, $i = 1, 2$,

$$\begin{aligned} &\mathsf{E}\exp\{\beta(Y_i^{(\varepsilon)}(t+u) - Y_i^{(\varepsilon)}(t))\} \\ &\quad = \begin{cases} (e^{\beta u_i^{(\varepsilon)}} p_1^{(\varepsilon)} + e^{-\beta u_i^{(\varepsilon)}} p_2^{(\varepsilon)})^{[(t+u)/\varepsilon]-[t/\varepsilon]}, & \text{if} \quad \varepsilon > 0, \\ e^{\beta\mu_i u + \frac{\beta^2\sigma_i^2 u}{2}}, & \text{if} \quad \varepsilon = 0. \end{cases} \end{aligned} \tag{29}$$

where $p_1^{(\varepsilon)} = p_{++}^{(\varepsilon)} + p_{+-}^{(\varepsilon)}$, $p_2^{(\varepsilon)} = p_{-+}^{(\varepsilon)} + p_{--}^{(\varepsilon)}$.

This makes it easy to check that condition $\mathbf{D_3}$ holds for processes $\vec{Y}^{(\varepsilon)}(t)$ for any $\beta' > \beta$.

Summarizing the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 3 holds, i.e., $\Phi(\mathcal{M}_{max,T}^{(\varepsilon)}) \to \Phi(\mathcal{M}_{max,T}^{(0)})$ as $\varepsilon \to 0$, for the bivariate exponential price processes with independent increments $\vec{S}^{(\varepsilon)}(t) = \exp\{\vec{Y}^{(\varepsilon)}(t)\}, t \in [0, T]$, if conditions $\mathbf{A_1}$, $\mathbf{A_4}$ hold for the corresponding payoff functions.

Let assume for simplicity that $\varepsilon = T/N$ and consider the partition $\Pi_\varepsilon = \langle t_0 = 0 < T_1 = \varepsilon < \cdots < t_{n-1} = (N-1)\varepsilon < t_N = T\rangle$ of interval $[0, T]$.

In this case the Markov chain $(n, \vec{Y}^{(\varepsilon)}(n\varepsilon))$, $n = 0, 1, \ldots$ is a bivariate binomial tree model with the initial node $(0, \vec{y}^{(0)})$ and $(n+1)^2$ nodes of the form $(n, (y_1^{(0)}, y_2^{(0)}) + ((2l_1 - n)\sqrt{\varepsilon}\sigma_1, (2l_2 - n)\sqrt{\varepsilon}\sigma_2))$, $l_1, l_2 = 0, 1, \ldots, n, i = 1, 2$ after $n \geq 1$ steps.

In the case of approximation of the continuous type option with maturity T by the corresponding discrete time model with time step $\varepsilon = T/N$ the corresponding tree has N steps, and therefore $(N+1)^2$ nodes after the last N-th step, $(N-1)^2$ nodes after $(N-1)$-th step, etc.

The standard backward procedure can be applied in order to find the optimal expected reward at moment 0 for the discrete time exponential bivariate binomial price process $\vec{S}^{(\varepsilon)}(n\varepsilon) = \exp\{\vec{Y}^{(\varepsilon)}(n\varepsilon)\}$. This optimal expected reward coincides, in this case, with the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ for the bivariate exponential price processes $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(\varepsilon)}(t)}$.

To estimate the difference $\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ we can use Theorem 1. In this case, $d(\Pi_\varepsilon) = \varepsilon$ and $\Delta_\beta(Y_i^{(\varepsilon)}(\cdot), \varepsilon, T) = \mathsf{E}e^{\beta|Y^{(\varepsilon)}_{1,i}|} - 1 \leq e^{\beta u_i^{(\varepsilon)}} - 1$, for $i = 1, 2$.

Theorem 1 yields in this case the following bound:

$$\begin{aligned}\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) &- \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) \\ &\leq L_2\varepsilon + L_3(e^{\beta u_1^{(\varepsilon)}} - 1 + e^{\beta u_2^{(\varepsilon)}} - 1)^{\frac{\beta-\gamma}{\beta}} \to 0 \text{ as } \varepsilon \to 0. \qquad (30)\end{aligned}$$

Thus, Theorem 3 guarantees that the optimal expected reward $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ converges under conditions $\mathbf{A_1}$, $\mathbf{A_4}$ to the reward functional $\Phi(\mathcal{M}^{(0)}_{max,T})$ for the bivariate geometric Brownian motion $\vec{S}^{(0)}(t) = e^{\vec{Y}^{(0)}(t)}$.

As an example of typical pay-off function, a linear combination of pay-off functions for option portfolio $g(t, \vec{s}) = e^{-rt}(a_1[s_1 - K_1]_+ + a_2[s_2 - K_2]_+)$ can be mentioned.

Let us also mention the model of exchange of assets with pay-off function $g(t, \vec{s}) = e^{-rt}(s_1 - s_2)$. Note that this is an example of pay-off function which is not nonnegative.

The optimal stopping strategies for this model were recently studied in Mishura and Shevchenko (2008).

In both cases, the payoff functions are continuous, do not depend on perturbation parameter ε, and have obviously a polynomial rate of growth. Thus, conditions $\mathbf{A_1}$ and $\mathbf{A_4}$ automatically hold.

Therefore, according to Theorem 3, the optimal expected reward functions for the described above bivariate binomial exponential model converge to the corresponding optimal expected reward functionals for the corresponding bivariate geometric Brownian motion.

It is useful to note that the results concerning bivariate binomial models admit an obvious generalization to the case of multivariate binomial models.

4. Binomial-trinomial approximations for reselling model

Let us continue consideration of reselling model introduced in Section 2.

In this model, there exists explicit solution to the system of stochastic differential equations (1) and (3) supplemented by the correlation relation (4). It is given by the following formulas,

$$\begin{cases} S(t) & = S(0)e^{\mu t + \sigma W_1(t)},\ t \geq 0 \\ \sigma(t) & = \sigma e^{\nu e^{-\alpha t} \int_0^t e^{\alpha s} dW_2(s)},\ t \geq 0, \end{cases} \tag{31}$$

where $\vec{W}(t) = (W_1(t), W_2(t)), t \geq 0$ is a bivariate Brownian motion which components are standard Brownian motions such that $\mathsf{E}W_1(1)W_2(1) = \rho$.

The process $(S(t), \sigma(t))$ is a Markov process.

Therefore, our object is the reward functional $\Phi(\mathcal{M}_T)$ for American type option with the pay-off function $e^{-rt}C(t, S, \sigma)$ for this bivariate Markov process $(S(t), \sigma(t))$.

The problem can be however reduced to the simpler case of a bivariate process, with independent increments.

Let us consider processes

$$S_1^{(0)}(t) = e^{\sigma W_1(t)}, t \geq 0,\ S_2^{(0)}(t) = e^{\nu e^{-\alpha T} \int_0^t e^{\alpha s} dW_2(s)}, t \geq 0. \tag{32}$$

In this case the vector process $\vec{S}^{(0)}(t) = (S_1^{(0)}(t), S_2^{(0)}(t)), t \geq 0$ is a bivariate continuous non-homogeneous exponential Gaussian process with independent increments.

Obviously, the filtration $\mathcal{F}_t = \sigma((S(s), \sigma(s)), s \leq t)$ generated by the vector process $(S(t), \sigma(t))$ coincides with the filtration $\mathcal{F}_t = \sigma((S_1^{(0)}(s), S_2^{(0)}(s)), s \leq t), t \geq 0$ generated by the bivariate process $\vec{S}^{(0)}(t)$.

Thus, the class $\mathcal{M}_T$ that includes all stopping times $0 \leq \tau \leq T$ that are Markov moments with respect to the filtration $\mathcal{F}_t, t \geq 0$, does not depend on which bivariate process is taken as a generator of this filtration.

Let us now define a pay-off function,

$$g(t, \vec{s}) = e^{-rt}C(t, S(0)e^{\mu t}s_1, \sigma s_2^{e^{\alpha(T-t)}}) \tag{33}$$

Note that this pay-off function does not depend of perturbation parameter ε, and its derivatives have not more than polynomial rates of growth. More precisely, condition $\mathbf{A_1}$ holds for this function with some constants $K_i, i = 1, \ldots, 5$ and the parameters $\gamma_0 = 2 + e^{2\alpha T}, \gamma_1 = 0$, and $\gamma_2 = e^{2\alpha T}$, and, therefore, $\gamma = 2 + e^{2\alpha T}$.

It readily follows from the remarks above that the reward functional,

$$\Phi(\mathcal{M}_T) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}e^{-r\tau}C(\tau, S(\tau), \sigma(\tau)) = \sup_{\tau \in \mathcal{M}_T} \mathsf{E}g(\tau, \vec{S}^{(0)}(\tau)) \tag{34}$$

Therefore, the reward functional $\Phi(\mathcal{M}_T)$ is an optimal expected reward for American type option with the payoff function $g(t, \vec{s})$ for this bivariate exponential process with independent increments $\vec{S}^{(0)}(t)$.

Let us now consider the corresponding bivariate log-price process $\vec{Y}^{(0)}(t) = (Y_1^{(0)}(t), Y_1^{(0)}(t)), t \geq 0$ with the components

$$Y_1^{(0)}(t) = \sigma W_1(t), t \geq 0, \ Y_2^{(0)}(t) = \nu e^{-\alpha T} \int_0^t e^{\alpha s} dW_2(s), t \geq 0. \qquad (35)$$

We approximate the process $\vec{Y}^{(0)}(t)$ by a bivariate binomial-trinomial sum-process $\vec{Y}^{(\varepsilon)}(t) = (Y_1^{(\varepsilon)}(t), Y_2^{(\varepsilon)}(t)), t \geq 0$ with components

$$Y_i^{(\varepsilon)}(t) = \sum_{1 \leq n \leq [t/\varepsilon]} Y_{n,i}^{(\varepsilon)}, \ t \geq 0, \ i = 1, 2. \qquad (36)$$

Here $\varepsilon > 0$ and $\vec{Y}_n^{(\varepsilon)} = (Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)}), n = 1, 2, \ldots$ are independent random vectors which take 6 values $(+u_1^{(\varepsilon)}, +u_2^{(\varepsilon)})$, $(+u_1^{(\varepsilon)}, 0)$, $(+u_1^{(\varepsilon)}, -u_2^{(\varepsilon)})$, $(-u_1^{(\varepsilon)}, +u_2^{(\varepsilon)})$, $(-u_1^{(\varepsilon)}, 0)$, and $(-u_1^{(\varepsilon)}, -u_2^{(\varepsilon)})$ with probabilities $p_{n,++}^{(\varepsilon)}$, $p_{n,+\cdot}^{(\varepsilon)}$, $p_{n,+-}^{(\varepsilon)}$, $p_{n,-+}^{(\varepsilon)}$, $p_{n,-\cdot}^{(\varepsilon)}$, and $p_{n,--}^{(\varepsilon)}$, respectively.

Let assume for simplicity that $\varepsilon = T/N$.

In order to fit the bivariate binomial-trinomial sum-processes $\vec{Y}^{(\varepsilon)}(t)$ to the limit bivariate process $\vec{Y}^{(0)}(t)$, we should fit expectations, variances, and covariance between components for random vectors $(Y_{n,1}^{(\varepsilon)}, Y_{n,2}^{(\varepsilon)})$ (whose values do not depend on n but the corresponding probabilities do depend on n) to the corresponding quantities for the increments $(\sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon)), \nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW_2(s))$ (which distributions do depend on n), for $n = 1, \ldots, N$.

Note that it is very important to make the jump values $u_1^{(\varepsilon)}$ and $u_2^{(\varepsilon)}$ independent on i since it automatically would provide the corresponding recombination condition for bivariate binomial-trinomial tree and the desirable quadratic rate of growth of number of nodes as function of the number of tree steps.

It is obvious that $\mathsf{E}\, \sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon)) = 0, \mathsf{V}ar\, \sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon)) = \sigma^2 \varepsilon$, and $\mathsf{E}\, \nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} \nu e^{\alpha s} dW_2(s) = 0$. Also,

$$\begin{aligned} \sigma_{n,\varepsilon}^2 &= \mathsf{V}ar\, \nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW_2(s) \\ &= \nu^2 e^{-2\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{2\alpha s} ds = \nu^2 e^{-2\alpha T} e^{2\alpha n\varepsilon} \frac{1 - e^{-2\alpha\varepsilon}}{2\alpha}. \end{aligned} \qquad (37)$$

and

$$\begin{aligned}\varrho_{n,\varepsilon} &= \mathsf{E}\,\sigma(W_1(n\varepsilon) - W_1((n-1)\varepsilon)) \cdot \nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} dW_2(s) \\ &= \rho\sigma\nu e^{-\alpha T} \int_{(n-1)\varepsilon}^{n\varepsilon} e^{\alpha s} ds = \sigma\rho\nu e^{-\alpha T} e^{\alpha n\varepsilon} \frac{1 - e^{-\alpha\varepsilon}}{\alpha}. \end{aligned} \tag{38}$$

The following system of $6N$ equations with $6N+2$ unknowns should be solved,

$$\begin{cases} \mathsf{E}[Y^{(\varepsilon)}_{1,1}] & = u^{(\varepsilon)}_1(2(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,+\cdot}) - 1) & = 0, \\ \mathsf{V}ar[Y^{(\varepsilon)}_{1,1}] & = (u^{(\varepsilon)}_1)^2 & = \sigma^2\varepsilon, \\ \mathsf{E}[Y^{(\varepsilon)}_{1,2}] & = u^{(\varepsilon)}_2(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} - p^{(\varepsilon)}_{n,+-} - p^{(\varepsilon)}_{n,--}) & = 0, \\ \mathsf{V}ar[Y^{(\varepsilon)}_{1,2}] & = (u^{(\varepsilon)}_2)^2(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,--}) & = \sigma^2_{n,\varepsilon}, \\ \mathsf{E}Y^{(\varepsilon)}_{1,1}Y^{(\varepsilon)}_{1,2} & = u^{(\varepsilon)}_1 u^{(\varepsilon)}_2(p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,--} - p^{(\varepsilon)}_{n,-+} - p^{(\varepsilon)}_{+-}) & = \varrho_{n,\varepsilon}, \\ & \quad p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,--} + p^{(\varepsilon)}_{n,+\cdot} + p^{(\varepsilon)}_{n,-\cdot} & = 1, \\ n = 1, \dots, N. \end{cases}$$

We can search solution $u^{(\varepsilon)}_2 = u\sqrt{\varepsilon}$ where u is the parameter under our control, due to the fact that the number of unknowns exceeds the number of equation.

The system above can be transformed to the following form:

$$\begin{cases} u^{(\varepsilon)}_1 & = \sigma\sqrt{\varepsilon}, \\ u^{(\varepsilon)}_2 & = u\sqrt{\varepsilon}, \\ p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,+\cdot} & = \frac{1}{2}, \\ p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+} & = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon}, \\ p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,--} & = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon} \\ p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,--} - p^{(\varepsilon)}_{n,-+} - p^{(\varepsilon)}_{+-} & = \frac{\varrho_{n,\varepsilon}}{\sigma u\varepsilon}, \\ p^{(\varepsilon)}_{n,+\cdot} + p^{(\varepsilon)}_{n,-\cdot} & = 1 - \frac{\sigma^2_{n,\varepsilon}}{u^2\varepsilon}, \\ n = 1, \dots, N. \end{cases}$$

and then to the following final form,

$$\begin{cases} u^{(\varepsilon)}_1 & = \sigma\sqrt{\varepsilon}, \\ u^{(\varepsilon)}_2 & = u\sqrt{\varepsilon}, \\ p^{(\varepsilon)}_{n,++} & = \frac{1}{4} + \frac{\varrho_{n,\varepsilon}}{4\sigma u\varepsilon} - \frac{1}{2}p^{(\varepsilon)}_{n,+\cdot}, \\ p^{(\varepsilon)}_{n,--} & = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon} - \frac{1}{4} - \frac{\varrho_{n,\varepsilon}}{4\sigma u\varepsilon} + \frac{1}{2}p^{(\varepsilon)}_{n,+\cdot}, \\ p^{(\varepsilon)}_{n,-+} & = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon} - \frac{1}{4} - \frac{\varrho_{n,\varepsilon}}{4\sigma u\varepsilon} + \frac{1}{2}p^{(\varepsilon)}_{n,+\cdot} \\ p^{(\varepsilon)}_{n,+-} & = \frac{1}{4} + \frac{\varrho_{n,\varepsilon}}{4\sigma u\varepsilon} - \frac{1}{2}p^{(\varepsilon)}_{n,+\cdot} \\ p^{(\varepsilon)}_{n,-\cdot} & = 1 - \frac{\sigma^2_{n,\varepsilon}}{u^2\varepsilon} - p^{(\varepsilon)}_{n,+\cdot}, \\ n = 1, \dots, N. \end{cases} \tag{39}$$

Let us take probability $p^{(\varepsilon)}_{n,+\cdot} = \frac{1}{2} + \frac{\varrho_{n,\varepsilon}}{2\sigma u \varepsilon} - \frac{\sigma^2_{n,\varepsilon}}{u^2 \varepsilon}$. It is easy to show that $\nu^2 e^{-2\alpha T} \leq \frac{\sigma^2_{n,\varepsilon}}{\varepsilon} \leq \nu^2 e^{-2\alpha T} e^{2\alpha T} = \nu^2$ and $\rho\sigma\nu e^{-\alpha T} \leq \frac{\varrho_{n,\varepsilon}}{\varepsilon} \leq \rho\sigma\nu e^{-\alpha T} e^{\alpha T} = \rho\sigma\nu$, for $n = 1, \dots N$. Thus, it is possible to choose u large enough such that the value of probability $0 < \frac{1}{2} + \frac{\rho\nu e^{-\alpha T}}{2u} - \frac{\nu^2}{u^2} \leq p^{(\varepsilon)}_{n,+\cdot} \leq \frac{1}{2} + \frac{\rho\nu}{2u} - \frac{\nu^2 e^{-2\alpha T}}{u^2} < 1$, for $n = 1, \dots N$.

Now, according to (39), the values $p^{(\varepsilon)}_{n,-+} = p^{(\varepsilon)}_{n,--} = 0$ while the values $p^{(\varepsilon)}_{n,++} = p^{(\varepsilon)}_{n,+-} = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon}$. It is also possible to choose u in such a way that $0 < \frac{\nu^2 e^{-2\alpha T}}{2u^2} \leq p^{(\varepsilon)}_{n,++} \leq \frac{\nu^2}{2u^2} < 1$, for $n = 1, \dots N$.

Finally, according (39), the value $p^{(\varepsilon)}_{n,-\cdot} = 1 - \frac{\sigma^2_{n,\varepsilon}}{u^2\varepsilon} - p^{(\varepsilon)}_{n,+\cdot} = \frac{1}{2} - \frac{\varrho_{n,\varepsilon}}{2\sigma u\varepsilon}$. It is also possible to choose u in such a way that $0 < \frac{1}{2} - \frac{\rho\nu}{2u} \leq p^{(\varepsilon)}_{n,-\cdot} \leq \frac{1}{2} - \frac{\rho\nu e^{-\alpha T}}{2u} < 1$, for $n = 1, \dots N$.

Thus, with the value of u chosen as described above we have the following values of probabilities,

$$\begin{cases} p^{(\varepsilon)}_{n,++} & = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon} & = \frac{\nu^2 e^{-2\alpha T}}{2u^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon}, \\ p^{(\varepsilon)}_{n,--} & = 0, & \\ p^{(\varepsilon)}_{n,-+} & = 0, & \\ p^{(\varepsilon)}_{n,+-} & = \frac{\sigma^2_{n,\varepsilon}}{2u^2\varepsilon} & = \frac{\nu^2 e^{-2\alpha T}}{2u^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon}, \\ p^{(\varepsilon)}_{n,+\cdot} & = \frac{1}{2} + \frac{\varrho_{n,\varepsilon}}{2\sigma u\varepsilon} - \frac{\sigma^2_{n,\varepsilon}}{u^2\varepsilon} & = \frac{1}{2} + \frac{\rho\nu e^{-\alpha T}}{2u} e^{\alpha n\varepsilon} \frac{1-e^{-\alpha\varepsilon}}{\alpha\varepsilon} - \frac{\nu^2 e^{-2\alpha T}}{u^2} e^{2\alpha n\varepsilon} \frac{1-e^{-2\alpha\varepsilon}}{2\alpha\varepsilon}, \\ p^{(\varepsilon)}_{n,-\cdot} & = \frac{1}{2} - \frac{\varrho_{n,\varepsilon}}{2\sigma u\varepsilon} & = \frac{1}{2} - \frac{\rho\nu e^{-\alpha T}}{2u} e^{\alpha n\varepsilon} \frac{1-e^{-\alpha\varepsilon}}{\alpha\varepsilon}, \\ & n = 1, \dots, N. & \end{cases}$$

By applying convergence theorems for vector sum-processes with independent increments given in Skorokhod (1964) it is possible to check that processes $\vec{Y}^{(\varepsilon)}(t), t \in [0, T]$ with parameters given in (39) weakly and J-converge to process $\vec{Y}^{(0)}(t), t \in [0, T]$ as $\varepsilon \to 0$.

Also, the moment generation function $\mathsf{E}\exp\{\beta(Y^{(\varepsilon)}_i(t+s) - Y^{(\varepsilon)}_i(t))\}$ exists for any $\beta \in \mathbb{R}_1$ and has an explicit form, namely, for $0 \leq t \leq t+s \leq T$, $i = 1, 2$,

$$\begin{aligned} &\mathsf{E}\exp\{\beta(Y^{(\varepsilon)}_1(t+s) - Y^{(\varepsilon)}_1(t))\} \\ &\quad = \begin{cases} (e^{\beta u^{(\varepsilon)}_1} p^{(\varepsilon)}_{1,+} + e^{-\beta u^{(\varepsilon)}_i} p^{(\varepsilon)}_{1,-})^{[(t+s)/\varepsilon]-[t/\varepsilon]}, & \text{if} \quad \varepsilon > 0, \\ e^{\frac{\beta^2\sigma^2 s}{2}}, & \text{if} \quad \varepsilon = 0. \end{cases} \end{aligned} \tag{40}$$

where $p^{(\varepsilon)}_{1,+} = p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,+\cdot}$, $p^{(\varepsilon)}_{2,-} = p^{(\varepsilon)}_{n,-+} + p^{(\varepsilon)}_{n,--} + p^{(\varepsilon)}_{n,-\cdot}$ (note that probabilities $p^{(\varepsilon)}_{1,+}$ and $p^{(\varepsilon)}_{2,-}$ do not depend on n), and

$$\begin{aligned} &\mathsf{E}\exp\{\beta(Y^{(\varepsilon)}_2(t+s) - Y^{(\varepsilon)}_2(t))\} \\ &\quad = \begin{cases} \prod_{[t/\varepsilon]+1}^{[(t+s)/\varepsilon]} (e^{\beta u^{(\varepsilon)}_2} p^{(\varepsilon)}_{n,2,+} + e^{-\beta u^{(\varepsilon)}_2} p^{(\varepsilon)}_{n,2,-} + p^{(\varepsilon)}_{n,2,\cdot}), & \text{if} \quad \varepsilon > 0, \\ e^{\frac{1}{2}\beta^2\nu^2 e^{-2\alpha T} \int_t^{t+s} e^{2\alpha v} dv}, & \text{if} \quad \varepsilon = 0. \end{cases} \end{aligned} \tag{41}$$

where $p^{(\varepsilon)}_{n,2,+} = p^{(\varepsilon)}_{n,++} + p^{(\varepsilon)}_{n,-+}$, $p^{(\varepsilon)}_{n,2,-} = p^{(\varepsilon)}_{n,+-} + p^{(\varepsilon)}_{n,--}$, $p^{(\varepsilon)}_{n,2,+} = p^{(\varepsilon)}_{n,+\cdot} + p^{(\varepsilon)}_{n,-\cdot}$.

Using formulas (40) and (41) it is readily possible to check that condition $\mathbf{D_3}$ holds for processes $\vec{Y}^{(\varepsilon)}(t)$ for any $\beta' > \beta$.

Summarizing the remarks above, one can conclude that the conditions and, therefore, the statement of Theorem 3 holds, i.e., $\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) \to \Phi(\mathcal{M}^{(0)}_{max,T})$ as $\varepsilon \to 0$, for the corresponding bivariate exponential price processes with independent increments $\vec{S}^{(\varepsilon)}(t) = \exp\{\vec{Y}^{(\varepsilon)}(t)\}, t \geq 0$.

Let us now consider the partition $\Pi_\varepsilon = \langle t_0 = 0 < t_1 = \varepsilon < \cdots < t_{N-1} = (N-1)\varepsilon < t_N = T \rangle$ of interval $[0, T]$.

In this case, the Markov chain $(n, \vec{Y}^{(\varepsilon)}(n\varepsilon))$, $n = 0, 1, \ldots$ is a bivariate binomial-trinomial tree model with the initial node $(0, (0, 0))$ and $(n + 1)(2n+1)$ nodes of the form $(n, ((2l_1 - n)\sqrt{\varepsilon}\sigma, l_2\sqrt{\varepsilon}u)), l_1, l_2 = 0, 1, \ldots, n, i = 1, 2$ after $n \geq 1$ steps.

In the case of approximation of the continuous type option with maturity T by the corresponding discrete time model with time step $\varepsilon = T/N$ the corresponding tree has N steps with $(N + 1)(2N + 1)$ nodes after the last N-th step, $((N - 1) + 1)(2(N - 1) + 1)$ nodes after $(N - 1)$-th step, etc.

The standard backward procedure can be applied in order to find the optimal expected reward at moment 0 for the discrete time exponential bivariate binomial-trinomial price process $\vec{S}^{(\varepsilon)}(n\varepsilon) = \exp\{\vec{Y}^{(\varepsilon)}(n\varepsilon)\}$. This optimal expected reward coincides, in this case, with the reward functional $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ for the bivariate exponential price processes $\vec{S}^{(\varepsilon)}(t) = e^{\vec{Y}^{(\varepsilon)}(t)}$.

To estimate the difference $\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ we can use Theorem 1. In this case, $d(\Pi_\varepsilon) = \varepsilon$ and $\Delta_\beta(Y^{(\varepsilon)}_1(\cdot), \varepsilon, T) = \mathsf{E}e^{\beta|Y^{(\varepsilon)}_{1,1}|} - 1 \leq e^{\beta\sigma\sqrt{\varepsilon}} - 1$ and $\Delta_\beta(Y^{(\varepsilon)}_2(\cdot), \varepsilon, T) = \max_{1 \leq n \leq N}(\mathsf{E}e^{\beta|Y^{(\varepsilon)}_{n,2}|} - 1) \leq e^{\beta u\sqrt{\varepsilon}} - 1$.

Theorem 1 yields in this case the following bound:

$$\begin{aligned} &\Phi(\mathcal{M}^{(\varepsilon)}_{max,T}) - \Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T}) \\ &\qquad \leq L_2\varepsilon + L_3(e^{\beta\sigma\sqrt{\varepsilon}} - 1 + e^{\beta u\sqrt{\varepsilon}} - 1)^{\frac{\beta-\gamma}{\beta}} \to 0 \text{ as } \varepsilon \to 0. \end{aligned} \tag{42}$$

Thus, Theorem 3 guarantees that the optimal expected reward $\Phi(\mathcal{M}^{(\varepsilon)}_{\Pi_\varepsilon,T})$ converges to the reward functional $\Phi(\mathcal{M}^{(0)}_{max,T})$ for the bivariate process with independent increments $\vec{S}^{(0)}(t) = e^{\vec{Y}^{(0)}(t)}$.

Therefore, according to Theorem 3 the optimal expected reward for the described above bivariate binomial-trinomial exponential model converges to the optimal expected reward functionals for the corresponding bivariate exponential Gaussian process with independent increments.

This conclusion completes our analysis of the reselling model.

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