

Goodness-of-fit test in a multivariate errors-in-variables model $AX = B$

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Received: 10 November 2016, Revised: 3 December 2016, Accepted: 4 December 2016,
Published online: 20 December 2016

Abstract We consider a multivariable functional errors-in-variables model $AX \approx B$, where the data matrices A and B are observed with errors, and a matrix parameter X is to be estimated. A goodness-of-fit test is constructed based on the total least squares estimator. The proposed test is asymptotically chi-squared under null hypothesis. The power of the test under local alternatives is discussed.

Keywords Goodness-of-fit test, local alternatives, multivariate errors-in-variables model, power of test, total least squares estimator

2010 MSC 62J05, 62H15, 65F20

1 Introduction

We study an overdetermined system of linear equations $AX \approx B$, which often occurs in the problems of dynamical system identification [10]. If matrices A and B are observed with additive uncorrelated errors of equal size, then the total least squares (TLS) method is used to solve the system [10].

In papers [3, 7, 9], under various conditions, the consistency of the TLS estimator \hat{X} is proven as the number m of rows in the matrix A is increasing, assuming that the true value A^0 of the input matrix is nonrandom. The asymptotic normality of the estimator is studied in [3] and [6].

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The model $AX \approx B$ with random measurement errors corresponds to the vector linear errors-in-variables model (EIVM). In [2], a goodness-of-fit test is constructed for a polynomial EIVM with nonrandom latent variable (i.e., in the *functional* case); the test can be also used in the *structural* case, where the latent variable is random with unknown probability distribution. A more powerful test in the polynomial EIVM is elaborated in [4].

In the paper [5], a goodness-of-fit test is constructed for the functional model $AX \approx B$, assuming that the error matrices \tilde{A} and \tilde{B} are independent and the covariance structure of \tilde{A} is known. In the present paper, we construct a goodness-of-fit test in a more common situation, where the total covariance structure of the matrices \tilde{A} and \tilde{B} is known up to a scalar factor. A test statistic is based on the TLS estimator \hat{X} . Under the null hypothesis, the asymptotic behavior of the test statistic is studied based on results of [6] and, under local alternatives, based on [9].

The present paper is organized as follows. In Section 2, we describe the observation model, introduce the TLS estimator, and formulate known results on the strong consistency and asymptotic normality of the estimator. In the next section, we construct the goodness-of-fit test and show that the proposed test statistic has an asymptotic chi-squared distribution with the corresponding number of degrees of freedom. The power of the test with respect to the local alternatives is studied in Section 4, and Section 5 concludes. The proofs are given in Appendix.

We use the following notation: $\|C\| = \sqrt{\sum_{i,j} c_{ij}^2}$ is the Frobenius norm of a matrix $C = (c_{ij})$, and I_p is the unit matrix of size p . The symbol \mathbf{E} denotes the expectation and acts as an operator on the total product of quantities, and \mathbf{cov} means the covariance matrix of a random vector. The upper index \top denotes transposition. In the paper, all the vectors are column ones. The bar means averaging over $i = 1, \dots, m$, for example, $\bar{a} := m^{-1} \sum_{i=1}^m a_i$, $\overline{ab^\top} := m^{-1} \sum_{i=1}^m a_i b_i^\top$. Convergence with probability one, in probability, and in distribution are denoted as $\xrightarrow{P1}$, \xrightarrow{P} , and \xrightarrow{d} , respectively. A sequence of random matrices that converges to zero in probability is denoted as $o_p(1)$, and a sequence of stochastically bounded random matrices is denoted as $O_p(1)$. The notation $\varepsilon \stackrel{d}{=} \varepsilon_1$ means that random variables ε and ε_1 have the same probability distribution. Positive constants that do not depend on the sample size m are denoted as *const*, so that equalities like $2 \cdot \text{const} = \text{const}$ are possible.

2 Observation model and total least squares estimator

2.1 The TLS problem

Consider the observation model

$$A^0 X^0 = B^0, \quad A = A^0 + \tilde{A}, \quad B = B^0 + \tilde{B}, \quad (2.1)$$

where $A^0 \in \mathbb{R}^{m \times n}$, $X^0 \in \mathbb{R}^{n \times d}$, and $B^0 \in \mathbb{R}^{m \times d}$. The matrices A and B contain the data, A^0 and B^0 are unknown nonrandom matrices, and \tilde{A} , \tilde{B} are the matrices of random errors.

We can rewrite model (2.1) in an implicit way. Introduce three matrices of size $m \times (n + d)$:

$$C^0 := [A^0 \ B^0], \quad \tilde{C} := [\tilde{A} \ \tilde{B}], \quad C := [A \ B]. \tag{2.2}$$

Then

$$C = C^0 + \tilde{C}, \quad C^0 \cdot \begin{bmatrix} X^0 \\ -I_d \end{bmatrix} = 0.$$

Let $A^\top = [a_1 \dots a_m]$, $B^\top = [b_1 \dots b_m]$, and we use similar notation for the rows of the matrices C , A^0 , B^0 , \tilde{A} , \tilde{B} , and \tilde{C} . Rewrite model (2.1) as a multivariate linear one:

$$X^{0\top} a_i^0 = b_i^0, \tag{2.3}$$

$$b_i = b_i^0 + \tilde{b}_i, \quad a_i = a_i^0 + \tilde{a}_i; \quad i = 1, \dots, m. \tag{2.4}$$

Throughout the paper, the following assumption holds about the errors $\tilde{c}_i = [\tilde{a}_i^\top \ \tilde{b}_i^\top]^\top$:

- (i) The vectors $\tilde{c}_i, i \geq 1$, are i.i.d. with zero mean, and, moreover,

$$\mathbf{cov}(\tilde{c}_1) = \sigma^2 I_{n+d}, \tag{2.5}$$

with unknown $\sigma > 0$.

Thus, the total error covariance structure is assumed to be known up to a scalar factor σ^2 , and the errors are uncorrelated with equal variances.

For model (2.1), the TLS problem lies in searching such disturbances $\Delta \hat{A}$ and $\Delta \hat{B}$ that minimize the sum of squared corrections

$$\min_{(X \in \mathbb{R}^{n \times d}, \Delta A, \Delta B)} (\|\Delta A\|^2 + \|\Delta B\|^2), \tag{2.6}$$

provided that

$$(A - \Delta A)X = B - \Delta B. \tag{2.7}$$

2.2 The TLS estimator and its consistency

It can happen that for certain random realization, the optimization problem (2.6)–(2.7) has no solution. In the latter case, we set $\hat{X} = \infty$.

Definition 1. The TLS estimator \hat{X} of the matrix parameter X^0 in the model (2.1) is a Borel-measurable function of the observed matrices A and B such that its values lie in $\mathbb{R}^{n \times d} \cup \{\infty\}$ and it provides a solution to problem (2.6)–(2.7) in case there exists a solution, and $\hat{X} = \infty$ otherwise.

We need the following conditions to provide the consistency of the estimator:

- (ii) $\mathbf{E} \|\tilde{c}_1\|^4 < \infty$.
- (iii) $\frac{1}{m} A^{0\top} A^0 \rightarrow V_A$ as $m \rightarrow \infty$, where V_A is a nonsingular matrix.

The next result on the strong consistency of the estimator follows, for example, from Theorem 4.3 in [9].

Theorem 2. *Assume conditions (i)–(iii). Then, with probability one, for all $m \geq m_0(\omega)$, the TLS estimator \hat{X} is finite, and, moreover, $\hat{X} \xrightarrow{P1} X^0$ as $m \rightarrow \infty$.*

Define the loss function $Q(X)$ as follows:

$$q(a, b; X) := (a^\top X - b^\top)(I_d + X^\top X)^{-1}(X^\top a - b), \tag{2.8}$$

$$Q(X) := \sum_{i=1}^m q(a_i, b_i; X), \quad X \in \mathbb{R}^{n \times d}. \tag{2.9}$$

It is known that the TLS estimator minimizes the loss function (2.9); see formula (24) in [7].

Introduce the following unbiased estimating function related to the elementary loss function (2.8):

$$s(a, b; X) = a(a^\top X - b^\top) - X(I_d + X^\top X)^{-1}(X^\top a - b)(a^\top X - b^\top). \tag{2.10}$$

Lemma 3. *Assume conditions (i)–(iii). Then, with probability one, for all $m \geq m_0(\omega)$, the TLS estimator \hat{X} is a solution to the equation*

$$\sum_{i=1}^m s(a_i, b_i; X) = 0, \quad X \in \mathbb{R}^{n \times d}.$$

In view of Theorem 2, the statement of Lemma 3 follows from Corollary 4(a) in [6].

2.3 Asymptotic normality of the estimator

We need further restrictions on the model. Recall that the augmented errors \tilde{c}_i were introduced in Section 2.2, and the vectors a_i^0 , \tilde{b}_i , and so on are those from model (2.3)–(2.4).

(iv) $\mathbf{E} \|\tilde{c}_1\|^{4+2\delta} < \infty$ for some $\delta > 0$;

(v) For δ from condition (iv), $\frac{1}{m^{1+\delta/2}} \sum_{i=1}^m \|a_i^0\|^{2+\delta} \rightarrow 0$ as $m \rightarrow \infty$.

Denote by $\tilde{c}_1^{(p)}$ the p th coordinate of the vector \tilde{c}_1 .

(vi) For all $p, q, r = 1, \dots, n + d$, we have $\mathbf{E} \tilde{c}_1^{(p)} \tilde{c}_1^{(q)} \tilde{c}_1^{(r)} = 0$.

Under assumptions (i) and (iv), condition (vi) holds, for example, in two cases: (a) when the random vector \tilde{c}_1 is symmetrically distributed, or (b) when the components of the vector \tilde{c}_1 are independent and, moreover, for each $p = 1, \dots, n + d$, the asymmetry coefficient of the random variable $\tilde{c}_1^{(p)}$ equals 0.

Introduce the following random element in the space of collections of five matrices:

$$W_i = (a_i^0 \tilde{a}_i^\top, a_i^0 \tilde{b}_i^\top, \tilde{a}_i \tilde{a}_i^\top - \sigma^2 I_n, \tilde{a}_i \tilde{b}_i^\top, \tilde{b}_i \tilde{b}_i^\top - \sigma^2 I_d). \tag{2.11}$$

The next statement on the asymptotic normality of the estimator follows from the proof of Theorem 8(b) in [6], where, instead of condition (vi), there was a stronger assumption that \tilde{c}_1 is symmetrically distributed, but the proof of Theorem 8(b) in [6] still works under the weaker condition (vi).

Theorem 4. *Assume conditions (i) and (iii)–(vi). Then:*

$$(a) \quad \frac{1}{\sqrt{m}} \sum_{i=1}^m W_i \xrightarrow{d} \Gamma = (\Gamma_1, \dots, \Gamma_5) \quad \text{as } m \rightarrow \infty, \quad (2.12)$$

where Γ is a Gaussian centered random element with matrix components,

$$(b) \quad \sqrt{m}(\hat{X} - X^0) \xrightarrow{d} V_A^{-1} \Gamma(X^0) \quad \text{as } m \rightarrow \infty, \quad (2.13)$$

$$\begin{aligned} \Gamma(X) := & \Gamma_1 X - \Gamma_2 + \Gamma_3 X - \Gamma_4 \\ & - X(\mathbf{I}_d + X^\top X)^{-1} (X^\top \Gamma_3 X - X^\top \Gamma_4 - \Gamma_4^\top X + \Gamma_5), \end{aligned} \quad (2.14)$$

where V_A is from condition (iii), and Γ_i is from condition (2.12).

Remark 5. Under the assumptions of Theorem 4, the components of random element (2.11) are uncorrelated, and therefore, the components of the limit element Γ are uncorrelated as well.

Let $f \in \mathbb{R}^{n \times 1}$. Under the conditions of Theorem 4, the convergence (2.13) implies that

$$\sqrt{m}(\hat{X} - X^0)^\top f \xrightarrow{d} N(0, S(X^0, f)), \quad (2.15)$$

$$S(X^0, f) = \mathbf{E} \Gamma^\top(X_0) V_A^{-1} f f^\top V_A^{-1} \Gamma(X_0). \quad (2.16)$$

Let a consistent estimator $\hat{f} = \hat{f}_m$ of the vector f be given. We want to construct a consistent estimator of matrix (2.16). The matrix $S(X^0, f)$ is expressed, for example, via the fourth moments of errors \tilde{c}_i , and those moments cannot be consistently estimated without additional assumptions on the error probability distribution. Therefore, an explicit expression for the latter matrix does not help to construct the desirable estimator. Nevertheless, we can construct something like the sandwich estimator [1, pp. 368–369].

The next statement on the consistency of the nuisance parameter estimators follows from the proof of Lemma 10 in [6]. Recall that the bar means averaging over the observations; see Section 1.

Lemma 6. *Assume the conditions of Theorem 4. Define the estimators:*

$$\hat{\sigma}^2 = \frac{1}{d} \text{tr}[(\overline{bb^\top} - 2\hat{X}^\top \overline{ab^\top} + \hat{X}^\top \overline{aa^\top} \hat{X})(\mathbf{I}_d + \hat{X}^\top \hat{X})^{-1}], \quad (2.17)$$

$$\hat{V}_A = \overline{aa^\top} - \hat{\sigma}^2 \mathbf{I}_n. \quad (2.18)$$

Then

$$\hat{\sigma}^2 \xrightarrow{P} \sigma^2, \quad \hat{V}_A \xrightarrow{P} V_A. \quad (2.19)$$

The next asymptotic expansion of the TLS estimator is presented in [6], formulas (4.10) and (4.11).

Lemma 7. *Under the conditions of Theorem 4, we have:*

$$\sqrt{m}(\hat{X} - X^0) = -V_A^{-1} \cdot \frac{1}{\sqrt{m}} \sum_{i=1}^m s(a_i, b_i; X^0) + o_p(1). \tag{2.20}$$

In view of Lemma 7, introduce the sandwich estimator $\hat{S}(\hat{f})$ of the matrix (2.16):

$$\hat{S}(\hat{f}) = \frac{1}{m} \sum_{i=1}^m s^\top(a_i, b_i; \hat{X}) \hat{V}_A^{-1} \hat{f} \hat{f}^\top \hat{V}_A^{-1} s(a_i, b_i; \hat{X}), \tag{2.21}$$

where the estimator \hat{V}_A is given in (2.18).

Theorem 8. *Let $f \in \mathbb{R}^{n \times 1}$, and let \hat{f} be a consistent estimator of this vector. Under the conditions of Theorem 4, the statistic $\hat{S}(\hat{f})$ is a consistent estimator of the matrix $S(X^0, f)$, that is, $\hat{S}(\hat{f}) \xrightarrow{P} S(X^0, f)$.*

Appendix contains the proof of this theorem and of all further statements.

3 Construction of goodness-of-fit test

For the observation model (2.4), we test the following hypotheses concerning the response b and the latent variable a^0 :

H₀ There exists such a matrix $X \in \mathbb{R}^{n \times d}$ that

$$\mathbf{E}(b - X^\top a^0) = 0, \quad \text{and} \tag{3.1}$$

H₁ For each matrix $X \in \mathbb{R}^{n \times d}$,

$$\mathbf{E}(b - X^\top a^0) \text{ is not identically zero.} \tag{3.2}$$

In fact, the null hypothesis means that the observation model (1.3)–(1.4) holds. Based on observations $a_i, b_i, i = 1, \dots, m$, we want to construct a test statistic to check this hypothesis. Let

$$T_m^0 := \frac{1}{m} \sum_{i=1}^m (b_i - \hat{X}^\top a_i) = \overline{b - \hat{X}^\top a}. \tag{3.3}$$

Lemma 9. *Under the conditions of Theorem 4,*

$$\sqrt{m}T_m^0 = \frac{1}{\sqrt{m}} \sum_{i=1}^m (\tilde{b}_i - X^{0\top} \tilde{a}_i) - \sqrt{m}(\hat{X} - X^0)^\top \overline{a^0} + o_p(1). \tag{3.4}$$

We need the following stabilization condition on the latent variable:

(vii) $\frac{1}{m} \sum_{i=1}^m a_i^0 \rightarrow \mu_a$ as $m \rightarrow \infty$ with $\mu_a \in \mathbb{R}^{n \times 1}$.

Lemma 10. Assume conditions (i) and (iii)–(vii). Then

$$\begin{aligned} \sqrt{m}T_m^0 &\xrightarrow{d} N(0, \Sigma_T), \\ \Sigma_T &= \sigma^2(1 - 2\mu_a^\top V_A^{-1}\mu_a)(I_d + X^{0\top}X^0) + S(X^0, \mu_a). \end{aligned} \quad (3.5)$$

Lemma 11. Assume the conditions of Lemma 10. Then:

- (a) A strong consistent estimator of the vector μ_a from condition (vii) is given by the statistic

$$\hat{\mu}_a := \bar{a} = \frac{1}{m} \sum_{i=1}^m a_i.$$

- (b) A consistent estimator of matrix (3.5) is given by the matrix statistic

$$\hat{\Sigma}_T := \hat{\sigma}^2(1 - 2\hat{\mu}_a^\top \hat{V}_A^{-1}\hat{\mu}_a)(I_d + \hat{X}^\top \hat{X}) + \hat{S}(\hat{\mu}_a), \quad (3.6)$$

where $\hat{\sigma}^2$ and \hat{V}_A are presented in (2.17) and (2.18), respectively, and $\hat{S}(\hat{\mu}_a)$ is matrix (2.21) with $\hat{f} = \hat{\mu}_a$.

To ensure the nonsingularity of the matrix Σ_T , we impose a final restriction on the observation model:

- (viii) There exists a finite matrix limit

$$S_a := \lim_{m \rightarrow \infty} \frac{1}{m} \sum_{i=1}^m (a_i^0 - \mu_a)(a_i^0 - \mu_a)^\top,$$

and, moreover, the matrix S_a is nonsingular.

Remark 12. Assume conditions (vii) and (viii). Then

$$\frac{1}{m} A^{0\top} A^0 = \frac{1}{m} \sum_{i=1}^m a_i^0 a_i^{0\top} \rightarrow V_A = S_a + \mu_a \mu_a^\top \quad \text{as } m \rightarrow \infty,$$

and V_A is nonsingular as a sum of positive definite and positive semidefinite matrices. Thus, condition (iii) is a consequence of assumptions (vii) and (viii).

Lemma 13. Assume conditions (i) and (iv)–(viii). Then:

- (a) Matrix (3.5) is positive definite.
 (b) With probability tending to one as $m \rightarrow \infty$, the symmetric matrix $\hat{\Sigma}_T$ is positive definite as well.

For $m \geq 1$ and ω from the underlying probability space Ω such that $\hat{\Sigma}_T$ is positive definite, we define the test statistic

$$T_m^2 = m \cdot \|\hat{\Sigma}_T^{-1/2} T_m^0\|^2. \quad (3.7)$$

Lemmas 10 and 11(b) imply the following convergence of the test statistic.

Theorem 14. Assume conditions (i) and (iv)–(viii). Then under hypothesis \mathbf{H}_0 , $T_m^2 \xrightarrow{d} \chi_d^2$ as $m \rightarrow \infty$.

Given a confidence level α , $0 < \alpha < 1/2$, let $\chi_{d\alpha}^2$ be the upper α -quantile of the χ_d^2 probability law, that is, $\mathbf{P}\{\chi_d^2 > \chi_{d\alpha}^2\} = \alpha$. Based on Theorem 14, we construct the following goodness-of-fit test with the asymptotic confidence probability $1 - \alpha$:

If $T_m^2 \leq \chi_{d\alpha}^2$, then we accept the null hypothesis,
and if $T_m^2 > \chi_{d\alpha}^2$, then we reject the null hypothesis.

4 Power of the test

Consider a sequence of models

$$\mathbf{H}_{1,m}: \quad b_i = X^\top a_i^0 + \frac{g(a_i^0)}{\sqrt{m}} + \tilde{b}_i, \quad a_i = a_i^0 + \tilde{a}_i, \quad i = 1, \dots, m. \quad (4.1)$$

Here $g: \mathbb{R}^n \rightarrow \mathbb{R}^d$ is a given nonlinear perturbation of the linear regression function.

For arbitrary function $f(a^0)$, denote the limit of averages

$$M(f(a^0)) = \lim_{m \rightarrow \infty} \overline{f(a^0)},$$

provided that the limit exists and is finite.

In order to study the behavior of the test statistic under local alternatives $\mathbf{H}_{1,m}$, we impose two restrictions on the perturbation function g :

(ix) There exist $M(g(a^0))$ and $M(g(a^0)a^{0\top})$.

(x) $\overline{\|g(a^0)\|^2} = o(m)$ as $m \rightarrow \infty$.

Under local alternatives $\mathbf{H}_{1,m}$, we ensure the weak consistency and asymptotic normality of the TLS estimator \hat{X} .

Lemma 15. Assume conditions (i) and (iv)–(x). Under local alternatives $\mathbf{H}_{1,m}$, we have:

(a) $\hat{X} \xrightarrow{P} X^0, \hat{\sigma}^2 \xrightarrow{P} \sigma^2$.

(b) $\sqrt{m}(\hat{X} - X^0) \xrightarrow{d} V_A^{-1}\Gamma(X^0) + V_A^{-1}M(a^0 g^\top(a^0))$ as $m \rightarrow \infty$,

where $\Gamma(X)$ is defined in (2.11), (2.12), and (2.14).

Lemma 16. Assume the conditions of Lemma 15. Then under local alternatives $\mathbf{H}_{1,m}$, we have:

(a) $\sqrt{m}T_m^0 \xrightarrow{d} N(C_T, \Sigma_T)$,

where Σ_T is given by (3.5), and

$$C_T := M(g(a^0)) - M(g(a^0)a^{0\top})V_A^{-1}\mu_a. \quad (4.2)$$

(b) The estimator $\hat{\Sigma}_T$ given in (3.6) tends in probability to the asymptotic covariance matrix Σ_T .

Now, we define the noncentral chi-squared distribution $\chi_d^2(\tau)$ with d degrees of freedom and the noncentrality parameter τ .

Definition 17. For $d \geq 1$ and $\tau \geq 0$, let $\chi_d^2(\tau) \stackrel{d}{=} \|N(\tau e, I_d)\|^2$, where $e \in \mathbb{R}^d$, $\|e\| = 1$, or, equivalently, $\chi_d^2(\tau) \stackrel{d}{=} (\gamma_1 + \tau)^2 + \sum_{i=2}^d \gamma_i^2$, where $\{\gamma_i\}$ are i.i.d. standard normal random variables.

Lemma 16 implies directly the following convergence.

Theorem 18. Assume conditions (i) and (iv)–(x). Then under local alternatives $\mathbf{H}_{1,m}$, we have:

$$T_m^2 \xrightarrow{d} \chi_d^2(\tau), \quad \tau := \|\Sigma_T^{-1/2} C_T\|, \tag{4.3}$$

where C_T is given in (4.2).

Theorem 18 makes it possible to find the asymptotic power of the test under local alternatives $\mathbf{H}_{1,m}$. It is evident that the asymptotic power is an increasing function of $\tau = \|\Sigma_T^{-1/2} C_T\|$. In other words, the larger τ , the more powerful the test.

5 Conclusion

We constructed a goodness-of-fit test for a multivariate linear errors-in-variables model, provided that the errors are uncorrelated with equal (unknown) variances and vanishing third moments. The latter moment assumption makes it possible to estimate consistently the asymptotic covariance matrix Σ_T of the statistic T_m^0 and construct the test statistic T_m^2 , which has the asymptotic χ_d^2 distribution under the null hypothesis. The local alternatives $\mathbf{H}_{1,m}$ are presented, under which the test statistic has the non-central $\chi_d^2(\tau)$ asymptotic distribution. The larger τ , the larger the asymptotic power of the test.

In future, we will try to construct, like in [5], a more powerful test using within a test statistic the exponential weight function

$$\omega_\lambda(a) = e^{\lambda^\top a}, \quad \lambda \in \mathbb{R}^{n \times 1}.$$

To this end, it is necessary to require the independence he terrors \tilde{b}_i and \tilde{a}_i and also the existence of exponential moments of the errors \tilde{a}_i . This is the price for a greater power of the test.

Appendix

Lemma 19. Let $r > 1$ be a fixed real number, and $\{\eta_k\}$ be an i.i.d. sequence with zero mean and finite moment $E|\eta_1|^r$. Assume also that a sequence $\{d_k\}$ of real numbers satisfies

$$\frac{1}{m^r} \sum_{k=1}^m |d_k|^r \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Then

$$\overline{d\eta} = \frac{1}{m} \sum_{k=1}^m d_k \eta_k \xrightarrow{P} 0. \tag{5.1}$$

Proof. Without of loss generality, we may and do assume that $1 < r < 2$. It suffices to check that the following three conditions from Theorem 5 in [8, Chap. VI] hold, which provide a criterion for the convergence (5.1):

$$\begin{aligned}
 \text{(a)} \quad & \sum_{k=1}^m \mathbf{P}\{|d_k \eta_k| > m\} \leq \sum_{k=1}^m \frac{\mathbf{E} |d_k \eta_k|^r}{m^r} = \frac{\mathbf{E} |\eta_1|^r}{m^r} \sum_{k=1}^m |d_k|^r \rightarrow 0 \text{ as } m \rightarrow \infty; \\
 \text{(b)} \quad & \frac{1}{m^2} \sum_{k=1}^m \mathbf{D}(d_k \eta_k \mathbf{I}(|d_k \eta_k| < m)) \leq \frac{1}{m^2} \sum_{k=1}^m \mathbf{E}(d_k^2 \eta_k^2 \mathbf{I}(|d_k \eta_k| < m)) \\
 & \leq \frac{1}{m^2} \sum_{k=1}^m \mathbf{E} |d_k \eta_k|^r \cdot m^{2-r} = \frac{\mathbf{E} |\eta_1|^r}{m^r} \sum_{k=1}^m |d_k|^r \rightarrow 0 \text{ as } m \rightarrow \infty; \\
 \text{(c)} \quad & \varepsilon_m := \frac{1}{m} \sum_{k=1}^m \mathbf{E}(d_k \eta_k \mathbf{I}(|d_k \eta_k| < m)) = -\frac{1}{m} \sum_{k=1}^m \mathbf{E}(d_k \eta_k \mathbf{I}(|d_k \eta_k| \geq m)), \\
 & |\varepsilon_m| \leq \frac{1}{m} \sum_{k=1}^m \mathbf{E} |d_k \eta_k|^r \cdot \frac{1}{m^{r-1}} = \frac{\mathbf{E} |\eta_1|^r}{m^r} \sum_{k=1}^m |d_k|^r \rightarrow 0 \text{ as } m \rightarrow \infty.
 \end{aligned}$$

By the mentioned theorem from [8] the presented bounds imply the desired convergence. □

The next statement is a version of the Lyapunov CLT.

Lemma 20. *Let $\{z_i\}$ be a sequence of independent centered random vectors in \mathbb{R}^p with $\mathbf{cov}(z) = \frac{1}{m} \sum_{i=1}^m \mathbf{cov}(z_i) \rightarrow S$ as $m \rightarrow \infty$. Assume also that, for some $\delta > 0$,*

$$\frac{1}{m^{1+\delta/2}} \sum_{i=1}^m \mathbf{E} \|z_i\|^{2+\delta} \leq \text{const}. \tag{5.2}$$

Then

$$\frac{1}{\sqrt{m}} \sum_{i=1}^m z_i \xrightarrow{d} N(0, S).$$

Proof of Theorem 8. (a) We have:

$$\begin{aligned}
 S(f) &:= \frac{1}{m} \sum_{i=1}^m s^\top(a_i, b_i; X^0) V_A^{-1} f f^\top V_A^{-1} s(a_i, b_i; X^0) \\
 &= (S(f) - \mathbf{E} S(f)) + \mathbf{E} S(f).
 \end{aligned} \tag{5.3}$$

In the proof of Theorem 8(a) in [6], the following expansion of the estimating function is used:

$$s(a_i, b_i; X^0) = W_{i1} X^0 - W_{i2} + W_{i3} X^0 - W_{i4} - X^0 (\mathbf{I}_d + X^{0\top} X^0)^{-1} \tag{5.4}$$

$$\times (X^{0\top} W_{i3} X^0 - X^{0\top} W_{i4} - W_{i4}^\top X^0 + W_{i5}), \tag{5.5}$$

where W_{ij} are the components of the matrix collection (2.11).

We show that the term in parentheses on the right-hand side of (5.3) tends to zero in probability. Taking into account expansion (5.5), we write down one of summands

of the expression $S(f)$:

$$L_m := \frac{1}{m} \sum_{i=1}^m X^{0\top} \tilde{a}_i a_i^{0\top} Z a_i^0 \tilde{a}_i^\top X^0, \quad Z := V_A^{-1} f f^\top V_A^{-1}. \quad (5.6)$$

Let us explain why

$$L_m - \mathbb{E} L_m \xrightarrow{\mathbb{P}} 0. \quad (5.7)$$

It suffices to consider the matrix

$$\tilde{L}_m := \frac{1}{m} \sum_{i=1}^m \tilde{a}_i a_i^{0\top} Z a_i^0 \tilde{a}_i^\top.$$

Up to a constant, its entries contain summands of the form

$$\frac{1}{m} \sum_{i=1}^m \tilde{a}_i^{(j)} a_i^{0(p)} a_i^{0(q)} \tilde{a}_i^{(r)}.$$

Applying Lemma 19 to the expression

$$\frac{1}{m} \sum_{i=1}^m a_i^{0(p)} a_i^{0(q)} (\tilde{a}_i^{(j)} \tilde{a}_i^{(r)} - \mathbb{E} \tilde{a}_i^{(j)} \tilde{a}_i^{(r)}), \quad (5.8)$$

we have $\mathbb{E}(\tilde{a}_i^{(j)} \tilde{a}_i^{(r)})^2 \leq \mathbb{E} \|\tilde{a}_i\|^4 < \infty$, and for δ from condition (v), we have:

$$\frac{1}{m^{1+\delta/2}} \sum_{i=1}^m |a_i^{0(p)} a_i^{0(q)}|^{1+\delta/2} \leq \frac{1}{m^{1+\delta/2}} \sum_{i=1}^m \|a_i^0\|^{2+\delta} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Thus, by Lemma 19 expression (5.8) tends to zero in probability. Then

$$\tilde{L}_m - \mathbb{E} \tilde{L}_m \xrightarrow{\mathbb{P}} 0,$$

whence we get (5.7).

In a similar way, other summands of $S(f)$ can be studied, and therefore,

$$S(f) - \mathbb{E} S(f) \xrightarrow{\mathbb{P}} 0.$$

Next, we verify directly the convergence

$$\mathbb{E} S(f) \rightarrow S(X^0, f) = \mathbb{E} \Gamma^\top (X^0) V_A^{-1} f f^\top V_A^{-1} \Gamma (X^0) \quad \text{as } m \rightarrow \infty.$$

Therefore, $S(f) \xrightarrow{\mathbb{P}} S(X^0, f)$.

(b) Without any problem, in view of Theorem 2 and the consistency of estimators \hat{V}_A and \hat{f} , the following convergences can be shown:

$$S(f) - \hat{S}(f) \xrightarrow{\mathbb{P}} 0, \quad \hat{S}(f) := \frac{1}{m} \sum_{i=1}^m s^\top (a_i, b_i; \hat{X}) \cdot Z \cdot s(a_i, b_i; \hat{X});$$

$$\hat{S}(f) - \hat{S}(\hat{f}) \xrightarrow{\mathbb{P}} 0.$$

Here Z is the matrix from relations (5.6).

The desired convergence follows from the convergences established in parts (a) and (b) of the proof. \square

Proof of Lemma 9. For model (2.3)–(2.4), we have:

$$\sqrt{m}T_m^0 = \frac{1}{\sqrt{m}} \sum_{i=1}^m (b_i^0 + \tilde{b}_i - \hat{X}^\top a_i^0 - \hat{X}^\top \tilde{a}_i^0) \quad (5.9)$$

$$= \frac{1}{\sqrt{m}} \sum_{i=1}^m (\tilde{b}_i - X^{0\top} \tilde{a}_i) - \sqrt{m}(\hat{X} - X^0)^\top \bar{a}^0 + rest, \quad (5.10)$$

$$\text{where } rest = -(\hat{X} - X^0)^\top \cdot \frac{1}{\sqrt{m}} \sum_{i=1}^m \tilde{a}_i = o_p(1) \cdot O_p(1) = o_p(1). \quad \square$$

Proof of Lemma 10. By Theorem 4(b),

$$\sqrt{m}(\hat{X} - X^0) = O_p(1).$$

Therefore, expansion (3.4) and condition (vii) imply that

$$\sqrt{m}T_m^0 = \frac{1}{\sqrt{m}} \sum_{i=1}^m (\tilde{b}_i - X^{0\top} \tilde{a}_i) + \sqrt{m}(\hat{X} - X^0)^\top \mu_a + o_p(1). \quad (5.11)$$

Next, by expansion (2.20) we get:

$$\sqrt{m}T_m^0 = \frac{1}{\sqrt{m}} \sum_{i=1}^m (\tilde{b}_i - X^{0\top} \tilde{a}_i + s^\top(a_i, b_i; X^0) V_A^{-1} \mu_a) + o_p(1). \quad (5.12)$$

The random vectors

$$z_i := \tilde{b}_i - X^{0\top} \tilde{a}_i + s^\top(a_i, b_i; X^0) V_A^{-1} \mu_a \quad (5.13)$$

satisfy condition (5.2) with the number δ from assumptions (iv) and (v). Let us find the variance–covariance matrix Σ_i of vector (5.13). We have

$$\Sigma_i = \mathbf{cov}(\tilde{b}_i - X^{0\top} \tilde{a}_i) + \mathbf{cov}(s^\top(a_i, b_i; X^0) V_A^{-1} \mu_a) + M + M^\top. \quad (5.14)$$

Here (see (2.11) and (5.5))

$$\begin{aligned} M &:= \mathbf{E} s^\top(a_i, b_i; X^0) V_A^{-1} \mu_a (\tilde{b}_i - \tilde{a}_i X^0) \\ &= \mathbf{E}(X^{0\top} \tilde{a}_i a_i^{0\top} - \tilde{b}_i a_i^{0\top}) V_A^{-1} \mu_a (\tilde{b}_i^\top - \tilde{a}_i^\top X^0); \\ M &= -X^{0\top} (\mathbf{E} \tilde{a}_i a_i^{0\top} V_A^{-1} \mu_a \tilde{a}_i^\top) X^0 - \mathbf{E} \tilde{b}_i a_i^{0\top} V_A^{-1} \mu_a \tilde{b}_i^\top \end{aligned} \quad (5.15)$$

$$= -a_i^{0\top} V_A^{-1} \mu_a \sigma^2 (\mathbf{I}_d + X^{0\top} X^0) = M^\top; \quad (5.16)$$

$$\mathbf{cov}(\tilde{b}_i - X^{0\top} \tilde{a}_i) = \sigma^2 (\mathbf{I}_d + X^{0\top} X^0);$$

$$\mathbf{cov}(s^\top(a_i, b_i; X^0) V_A^{-1} \mu_a) = \mathbf{E} s^\top(a_i, b_i; X^0) \cdot Z \cdot s(a_i, b_i; X^0).$$

Then

$$\begin{aligned} \Sigma_T &:= \lim_{m \rightarrow \infty} \frac{1}{m} (\Sigma_1 + \dots + \Sigma_m) \\ &= \sigma^2 (\mathbf{I}_d + X^{0\top} X^0) + S(X^0, \mu_a) - 2\mu_a^\top V_A^{-1} \mu_a \sigma^2 (\mathbf{I}_d + X^{0\top} X^0), \end{aligned}$$

and this coincides with the right-hand side of equality (3.5).

Finally, the desired convergence follows from expansion (5.12) by Lemma 20 and Slutsky’s lemma. \square

Proof of Lemma 11. The convergence $\hat{\mu}_a \xrightarrow{P1} \mu_a$ is established by SLLN. The convergence

$$\hat{\Sigma}_T \xrightarrow{P} \Sigma_T$$

follows from Theorem 8 (the role of f and \hat{f} is played by μ_a and $\hat{\mu}_a$, respectively) and the consistency of estimators $\hat{\sigma}^2$, $\hat{\mu}_a$, and \hat{V}_A . \square

Proof of Lemma 13. (a) Hereafter, for symmetric matrices A and B , notation $A \geq B$ ($A > B$) means that the matrix $A - B$ is positive semidefinite (positive definite).

Condition (vi) ensures the independence of the matrix components Γ_i in relation (2.12). Therefore,

$$S(X^0, \mu_a) \geq \mathbf{cov}((X^{0\top} \tilde{a}_i - \hat{b}_i) \mu_a^\top V_A^{-1} \mu_a) = \sigma^2 (\mu_a^\top V_A^{-1} \mu_a)^2 (\mathbf{I}_d + X^{0\top} X^0).$$

From equality (3.5) we have

$$\Sigma_T \geq \sigma^2 (1 - \mu_a^\top V_A^{-1} \mu_a)^2 \cdot \mathbf{I}_d. \tag{5.17}$$

By condition (viii), $V_A > \mu_a \mu_a^\top$. In the case $\mu_a = 0$, we get $\Sigma_T \geq \sigma^2 \mathbf{I}_d > 0$, and in the case $\mu_a \neq 0$, we put $z = V_A^{-1} \mu_a$ and obtain:

$$z^\top V_A z = \mu_a^\top V_A^{-1} \mu_a > (\mu_a^\top z)^2 = (\mu_a^\top V_A^{-1} \mu_a)^2;$$

thus, $1 > \mu_a^\top V_A^{-1} \mu_a$, and inequality (5.17) implies $\Sigma_T > 0$.

Statement (b) follows from statement (a) and Lemma 11(b). \square

Proof of Lemma 15. (a) The local alternative (4.1) is corresponding to the perturbation matrix

$$G^0 := \begin{bmatrix} g^\top(a_1^0) \\ \vdots \\ g^\top(a_m^0) \end{bmatrix}.$$

Model (4.1) can be rewritten as a perturbed model (2.1),

$$A^0 X^0 = B^0, \quad A = A^0 + \tilde{A}, \quad B^{per} := B^0 + \frac{1}{\sqrt{m}} G^0 + \tilde{B}, \tag{5.18}$$

or as a perturbed model (2.2),

$$C^0 = [A^0 \ B^0], \quad \tilde{C} = [\tilde{A} \ \tilde{B}], \quad C^{per} := [A \ B^{per}], \tag{5.19}$$

$$C^0 \cdot \begin{bmatrix} X^0 \\ -\mathbf{I}_d \end{bmatrix} = 0. \tag{5.20}$$

Introduce the symmetric matrix

$$N = C^{0\top} C^0 + \lambda_{\min}(A^{0\top} A^0) \mathbf{I}_{n+d}.$$

Due to condition (iii), as $m \rightarrow \infty$,

$$N = mN^0 + o(m), \quad N^0 > 0. \tag{5.21}$$

Consider two matrices of size $(n + d) \times (n + d)$:

$$M_1 = N^{-1/2} C^{0\top} (C^{per} - C^0) N^{-1/2}, \quad (5.22)$$

$$M_2 = N^{-1/2} ((C^{per} - C^0)^\top (C^{per} - C^0) - \sigma^2 m \mathbf{I}_{n+d}) N^{-1/2}. \quad (5.23)$$

In view of the proof of Theorem 4.1 in [9], for the convergence

$$\hat{X} \xrightarrow{P} X^0, \quad (5.24)$$

it suffices to show that, as $m \rightarrow \infty$,

$$M_1 \xrightarrow{P} 0, \quad M_2 \xrightarrow{P} 0, \quad (5.25)$$

or taking into account (5.21), that

$$M'_1 := \frac{1}{m} C^{0\top} \tilde{C} + \frac{1}{m} C^{0\top} \cdot \frac{1}{\sqrt{m}} G^0 \xrightarrow{P} 0, \quad (5.26)$$

$$M'_2 := \frac{1}{m} \left(\left(\tilde{C} + \frac{1}{\sqrt{m}} [0 \ G^0] \right)^\top \left(\tilde{C} + \frac{1}{\sqrt{m}} [0 \ G^0] \right) - \sigma^2 \mathbf{I}_{n+d} \right) \xrightarrow{P} 0. \quad (5.27)$$

We study the most interesting summands, those that contain G^0 (the convergence of other summands was shown in the proof of Theorem 4.1 in [9]). We have

$$M''_1 := \frac{1}{m^{3/2}} C^{0\top} G^0 = \frac{1}{m^{3/2}} \begin{bmatrix} A^{0\top} G^0 \\ X^{0\top} A^{0\top} G^0 \end{bmatrix},$$

and due to condition (ix), as $m \rightarrow \infty$,

$$\frac{1}{m^{3/2}} A^{0\top} G^0 = \frac{1}{m^{3/2}} \sum_{i=1}^m a_i^0 g^\top(a_i^0) = \frac{O(1)}{m^{1/2}} \rightarrow 0, \quad M''_1 \rightarrow 0. \quad (5.28)$$

Next, by condition (x),

$$M''_2 := \frac{1}{m^2} G^{0\top} G^0 = \frac{1}{m^2} \sum_{i=1}^m g(a_i^0) g^\top(a_i^0), \quad (5.29)$$

$$\|M''_2\| \leq \frac{const}{m^2} \sum_{i=1}^m \|g(a_i^0)\|^2 \rightarrow 0 \quad \text{as } m \rightarrow \infty. \quad (5.30)$$

Finally,

$$M''_2''' := \frac{1}{m^{3/2}} \tilde{C}^\top G^0 = \frac{1}{m^{3/2}} \sum_{i=1}^m \tilde{c}_i g^\top(a_i^0), \quad (5.31)$$

$$\mathbb{E} \|M''_2'''\|^2 \leq \frac{const}{m^3} \sum_{i=1}^m \|g(a_i^0)\|^2 \rightarrow 0 \quad \text{as } m \rightarrow \infty, \quad M''_2''' \xrightarrow{P} 0. \quad (5.32)$$

We established the convergence in probability for the summands from (5.26) and (5.27) that contain the perturbation G^0 . Therefore, (5.26) and (5.27) are satisfied, relation (5.25) is satisfied as well, and the results of [9] imply convergence (5.24).

The consistency of the estimator $\hat{\sigma}^2$ under local alternatives $\mathbf{H}_{1,m}$ is established by formula (2.17) and boils down to the consistency of $\hat{\sigma}^2$ under the null hypothesis: the consistency of \hat{X} has been proven already, and, moreover,

$$\overline{b^{per}b^{per,\top}} = \frac{1}{m} \sum_{i=1}^m \left(b_i + \frac{1}{\sqrt{m}} g(a_i^0) \right) \left(b_i + \frac{1}{\sqrt{m}} g(a_i^0) \right)^\top = \frac{1}{m} \sum_{i=1}^m b_i b_i^\top + o_p(1), \tag{5.33}$$

$$\overline{ab^{per,\top}} = \overline{ab^\top} + o_p(1). \tag{5.34}$$

(b) After we established the consistency of \hat{X} under alternatives $\mathbf{H}_{1,m}$, we find an expansion similar to (2.20):

$$\sqrt{m}(\hat{X} - X^0) = -V_A^{-1} \frac{1}{\sqrt{m}} \sum_{i=1}^m s(a_i, b_i^{per}; X^0) + o_p(1) \tag{5.35}$$

$$= -V_A^{-1} \frac{1}{\sqrt{m}} \sum_{i=1}^m s(a_i, b_i; X^0) - V_A^{-1} \frac{1}{\sqrt{m}} \sum_{i=1}^m s_i^{per} + o_p(1). \tag{5.36}$$

Conditions (ix) and (x) ensure that, for perturbations s_i^{per} of the estimating function, we have (the main contribution to s_i^{per} is made by a linear summand ab^\top from (2.10)):

$$\frac{1}{\sqrt{m}} \sum_{i=1}^m s_i^{per} = -\frac{1}{m} \sum_{i=1}^m a_i^0 g^\top(a_i^0) + o_p(1). \tag{5.37}$$

Lemma 7, Theorem 4, and formulae (5.36) and (5.37) imply the desired convergence of the normalized TLS estimator. \square

Proof of Lemma 16. (a) Under the local alternatives, we have:

$$\sqrt{m}T_m^0|_{\mathbf{H}_{1,m}} = \sqrt{m}T_m^0|_{\mathbf{H}_0} + M(g(a^0)) - \sqrt{m}(\hat{X}|_{\mathbf{H}_{1,m}} - \hat{X}|_{\mathbf{H}_0})^\top \mu_a + o_p(1).$$

Expansions (2.20), (5.36), and (5.37) imply that

$$\sqrt{m}(\hat{X}|_{\mathbf{H}_{1,m}} - \hat{X}|_{\mathbf{H}_0}) \xrightarrow{P} V_A^{-1} \cdot M(a^0 g^\top(a^0)).$$

Then, by Lemma 10 and Slutsky’s lemma,

$$\sqrt{m}T_m^0|_{\mathbf{H}_{1,m}} \xrightarrow{d} N(C_T, \Sigma_T), \tag{5.38}$$

$$C_T = M(g(a^0)) - M(g(a^0)a^{0\top}) \cdot V_A^{-1} \mu_a. \tag{5.39}$$

(b) Under the local alternatives, the estimators $\hat{\sigma}^2$, $\hat{\mu}_a$, \hat{V}_A , and \hat{X} are still consistent. Moreover,

$$\hat{S}(\hat{\mu}_a) = \frac{1}{m} \sum_{i=1}^m s^\top(a_i, b_i^{per}; \hat{X}) \hat{V}_A^{-1} \hat{\mu}_a \hat{\mu}_a^\top \hat{V}_A^{-1} s(a_i, b_i^{per}; \hat{X}) \tag{5.40}$$

converges in probability to $S(X^0, \mu_a)$ because expression (5.40) does not involve terms linear in b_i^{per} , and the perturbation of the vectors b_i does not modify the asymptotic behavior of $\hat{S}(\hat{\mu}_a)$ in transition from \mathbf{H}_0 to the local alternatives.

Thus, estimator (3.6) does converge in probability to matrix (3.5). \square

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