
Volume 98 Number 3 March 2007 ISSN 0047-259X
ELSEVIER
Journal of
Multivariate Analysis
JMVA
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
Journal of Multivariate Analysis 98 (2007) 588 – 624
Journal of Multivariate Analysis
www.elsevier.com/locate/jmva

On the conic section fitting problem

Sergiy Shklyar[a], Alexander Kukush[a,*], Ivan Markovsky[b], Sabine Van Huffel[b]

[a] Kiev National Taras Shevchenko University, Vladimirskaya st. 64, 01033, Kiev, Ukraine
[b] K.U.Leuven, ESAT-SCD, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium

Received 30 May 2005
Available online 31 January 2006

Abstract

Adjusted least squares (ALS) estimators for the conic section problem are considered. Consistency of the translation invariant version of ALS estimator is proved. The similarity invariance of the ALS estimator with estimated noise variance is shown. The conditions for consistency of the ALS estimator are relaxed compared with the ones of the paper Kukush et al. [Consistent estimation in an implicit quadratic measurement error model, *Comput. Statist. Data Anal.* 47(1) (2004) 123–147].

AMS 1991 subject classification: 62H35; 62F12; 62J02; 62J10; 62P12; 62P30

Keywords: Adjusted least squares; Conic fitting; Consistent estimator; Ellipsoid fitting; Quadratic measurement error model

1. Introduction

The problem considered in this paper is to estimate a hypersurface of the second order that fits a sample of points $x_1, x_2, \ldots, x_m$ in $\mathbb{R}^n$. A second order surface in $\mathbb{R}^n$ is described by the equation

$$x^\top Ax + b^\top x + d = 0. \tag{1}$$

Without loss of generality, one can assume the matrix A to be symmetric. Let $\mathbb{S}$ be a set of real $n \times n$ symmetric matrices. The set of all the triples (A, b, d) is $\mathbb{V} := \mathbb{S} \times \mathbb{R}^n \times \mathbb{R}$.

* Corresponding author. Fax. +380 44 2590392.
E-mail addresses: shklyar@mail.univ.kiev.ua (S. Shklyar), alexander_kukush@univ.kiev.ua (A. Kukush), ivan.markovsky@esat.kuleuven.ac.be (I. Markovsky), sabine.vanhuffel@esat.kuleuven.ac.be (S. Van Huffel).

doi:10.1016/j.jmva.2005.12.003

Consider a measurement error model. Assume that $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ lie on the true surface $\{x \mid x^\top \bar{A} x + \bar{b}^\top x + \bar{d} = 0\}$. In Section 4 we consider the structural case where $\bar{x}_1, \ldots, \bar{x}_m$ is an independent identically distributed sequence, while in the rest of the paper the model is functional, i.e., $\bar{x}_1, \ldots, \bar{x}_m$ are nonrandom. The true values are observed with errors, which give the measurements $x_1, \ldots, x_m$. The measurement errors are supposed to be identically distributed normal variables, the variance of which is either specified or unknown. The parameters of the true surface are parameters of interest. The conic section estimation problem arises in computer vision and meteorology, see [4] and [5].

We use the word "conic" in a very wide sense. Any set that can be defined by Eq. (1) is referred to as "conic". "The true conic" is neither the entire space $\mathbb{R}^n$ nor a subset of a hyperplane, and our conditions ensure that.

Consider the ordinary least squares (OLS) estimator, which is defined by the minimization of the loss function

$$Q_{\mathrm{ols}}(A, b, d) := \sum_{l=1}^{m} (x_l^\top A x_l + b^\top x_l + d)^2.$$

It is easy to compute but inconsistent in the errors-in-variables setup.

The orthogonal regression estimator is inconsistent as well, though it has smaller asymptotic bias [2, Example 3.2.4].

To reduce the asymptotic bias, the renormalization procedure can be used, see [4]. In [5] an adjusted loss function $Q_{\mathrm{als}}(\beta)$ is defined implicitly via the equation

$$\mathbb{E} Q_{\mathrm{als}}(A, b, d) = \sum_{l=1}^{m} (\bar{x}_l^\top A \bar{x}_l + b^\top \bar{x}_l + d)^2,$$

and consistency of the resulting ALS estimator is proved. A computational algorithm and a simulation study for the method of [5] are given in [7].

The ALS estimator with known error variance is not translation-invariant. In this paper we propose a translation-invariant modification of the ALS estimator (TALS estimator). Its consistency is shown. The translation invariance of the ALS estimator with estimated error variance is proved as well, and the conditions for consistency of the estimator are relaxed.

We propose a definition of invariance of an estimator. By appropriate choice of the parameter space and the estimation space this definition can be deduced from the definition of equivariance given in [6, Section 3.2].

The Euclidean norm of a vector $x = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ is denoted by $\|x\| := \sqrt{\sum_{i=1}^d x_i^2}$. If $A = (a_{i,j})$ is $m \times n$ matrix, $\|A\| := \max_{\|x\| \leqslant 1} \|Ax\|$, while $\|A\|_F := \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$ is its Frobenius norm. The rank of the matrix A is denoted by $\mathrm{rk} A$. If $m = n$, then $\mathrm{tr}\, A := \sum_{i=1}^m a_{ii}$ is the trace of A.

As a direct sum of three Euclidean spaces, $\mathbb{V}$ is a Euclidean space with inner product

$$\langle (A_1, b_1, d_1),\ (A_2, b_2, d_2) \rangle := \mathrm{tr}(A_1 A_2) + b_1^\top b_2 + d_1 d_2.$$

The induced norm is

$$\|(A, b, d)\| := \sqrt{\|A\|_F^2 + \|b\|^2 + d^2}.$$

The dimension of the space $\mathbb{V}$ is

$$n_\beta = \frac{n(n+1)}{2} + n + 1 = \frac{(n+1)(n+2)}{2}.$$

Construct an orthonormal basis of $\mathbb{V}$. For $n = 2$, the six triples

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0\right), \quad \left(\begin{pmatrix} 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0\right), \quad \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 0\right),$$
$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, 0\right), \quad \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, 0\right), \quad \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, 1\right)$$

form an orthonormal basis of $\mathbb{V}$. For an arbitrary $n \geqslant 1$ the set $\{\mathfrak{b}_1, \mathfrak{b}_2, \ldots, \mathfrak{b}_{n_\beta}\}$ is an orthonormal basis of $\mathbb{V}$, where

$$\begin{aligned}
\mathfrak{b}_{\frac{(i-1)i}{2}+j} &:= \left(\frac{e_i e_j^\top + e_j e_i^\top}{\sqrt{2}}, 0, 0\right), \quad 1 \leqslant j < i \leqslant n, \\
\mathfrak{b}_{\frac{i(i+1)}{2}} &:= (e_i e_i^\top, 0, 0), \quad 1 \leqslant i \leqslant n, \\
\mathfrak{b}_{\frac{n(n+1)}{2}+i} &:= (0, e_i, 0), \quad 1 \leqslant i \leqslant n, \\
\mathfrak{b}_{\frac{(n+1)(n+2)}{2}} &:= (0, 0, 1),
\end{aligned}$$

and $e_i := (0_1, \ldots, 0, 1_i, 0, \ldots, 0_n)^\top$ is the ith vector of the standard basis in $\mathbb{R}^n$.

Let $[\beta]$ be the vector of coordinates of $\beta \in \mathbb{V}$. Then $[\beta] = ([\beta]_1, \ldots, [\beta]_{n_\beta})^\top$ with $[\beta]_i := \langle \beta, \mathfrak{b}_i \rangle$ and $\beta = \sum_{i=1}^{n_\beta} [\beta]_i \mathfrak{b}_i$.

If Ψ is a linear operator on $\mathbb{V}$, then its matrix is denoted by $[\Psi]$. One has $[\Psi\beta] = [\Psi][\beta]$ for any $\beta \in \mathbb{V}$. The i, jth entry of $[\Psi]$ is equal to $[\Psi]_{ij} := \langle \Psi \mathfrak{b}_j, \mathfrak{b}_i \rangle$.

The ordered eigenvalues of a symmetric $d \times d$ matrix A are denoted by $\lambda_1(A) \leqslant \lambda_2(A) \leqslant \cdots \leqslant \lambda_d(A)$. We also use the notation $\lambda_{\min}(A) := \lambda_1(A)$, $\lambda_{\max}(A) := \lambda_d(A)$. Note that $\lambda_2(A) = \lambda_1(A)$ if the minimal eigenvalue is multiple.

If Ψ is a self-adjoint operator on $\mathbb{V}$, then $\|\Psi\| := \max_{\|\beta\|=1} \|\Psi\beta\|$ is its norm, and $\lambda_1(\Psi) \leqslant \lambda_2(\Psi) \leqslant \cdots \leqslant \lambda_{n_\beta}(\Psi)$ are its eigenvalues. Again, $\lambda_{\min}(\Psi) := \lambda_1(\Psi)$.

There is a natural one-to-one correspondence between self-adjoint operators, quadratic forms and symmetric matrices.

We occasionally omit the sample size in the notation. Estimates ($\hat{\beta}$, $\widehat{D}$) and variables denoted by letters Q, Ψ, S, s, with and without bars, with different subscripts, are defined for a fixed sample size m. The sequence of events $\{P_m,\ m \geqslant 1\}$ is said to occur eventually if

$$\mathbb{P}\left(\bigcup_{l=1}^{\infty} \bigcap_{m=l}^{\infty} P_m\right) = 1.$$

In Section 2 the implicit quadratic errors-in-variables model is described and the estimates are defined for both the case of specified and unknown variance. For the case of unknown variance, the ALS estimators of the surface and of the variance are studied in Section 3. The conditions for consistency of the estimators relax the assumptions of [5, Theorem 9], namely we do not assume the contrast condition (vi) from [5, p. 134]. The conditions for consistency in the structural model are given in Section 4. In Section 5 the invariance of the estimates is shown, and Section 6 concludes. Some auxiliary proofs are moved to Appendices A and B. In Appendix C we introduce the concepts which are used to derive bounds for perturbations of generalized eigenvectors.

2. The model and the estimates

2.1. The model

Consider a true conic in $\mathbb{R}^n$ defined by the equation

$$x^\top \bar{A} x + \bar{b}^\top x + \bar{d} = 0,$$

with parameters $\bar{\beta} := (\bar{A}, \bar{b}, \bar{d}) \in \mathbb{V}$. Assume that $\bar{\beta} \neq 0$. The parameters can be chosen, such that

$$\|\bar{A}\|_F^2 + \|\bar{b}\|^2 + \bar{d}^2 = 1. \tag{2}$$

Let nonrandom vectors $\bar{x}_1, \bar{x}_2, \ldots$ belong to the true conic:

$$\bar{x}_l^\top \bar{A} \bar{x}_l + \bar{b}^\top \bar{x}_l + \bar{d} = 0, \quad l = 1, 2, \ldots \tag{3}$$

The vectors belonging to the true surface are observed with errors. Let x_l be the measurement of $\bar{x}_l$, and $\tilde{x}_l$ be an error, i.e.

$$x_l = \bar{x}_l + \tilde{x}_l. \tag{4}$$

Let measurement errors satisfy the following conditions:

(i) $\tilde{x}_1, \tilde{x}_2, \ldots$ are totally independent,
(ii) $\tilde{x}_l$ is a normal vector, $\tilde{x}_l \sim N(0, \sigma^2 I)$, $\sigma > 0$.

Hereafter I is an identity matrix.

The specified model is a functional homoscedastic measurement error model, given in an implicit form. As usual in errors-in-variables setting 'functional' means that the true vectors $\bar{x}_1, \bar{x}_2, \ldots$ are nonrandom.

Let m be the sample size. The measurements $x_1, x_2, \ldots, x_m$ are observed. $\bar{\beta}$ and σ^2 are parameters of the model and $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ are nuisance parameters. Initially the parameter σ^2 is supposed to be known, but later on we will consider the case of unknown σ^2 as well.

2.2. Definition of the estimates

In this subsection the sample size m is fixed.

2.2.1. OLS estimator

The elementary OLS loss function is

$$q_{\text{ols}}((A, b, d), x) := (x^\top A x + b^\top x + d)^2, \quad (A, b, d) \in \mathbb{V}, \ \ x \in \mathbb{R}^n.$$

Let

$$\bar{Q}_{\text{ols}}(\beta) := \sum_{l=1}^{m} q_{\text{ols}}(\beta, \bar{x}_l), \quad \beta \in \mathbb{V}.$$

$\bar{Q}_{\text{ols}}(\beta)$ is a positive semidefinite quadratic form on the space $\mathbb{V}$. Equality $\bar{Q}_{\text{ols}}(A, b, d) = 0$ holds true if and only if all vectors $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ belong to the conic $\{x \in \mathbb{R}^n \mid x^\top Ax + b^\top x + d = 0\}$. By (3), $\bar{Q}_{\text{ols}}(\bar{\beta}) = 0$.

Let

$$Q_{\text{ols}}(\beta) := \sum_{l=1}^{m} q_{\text{ols}}(\beta, x_l), \quad \beta \in \mathbb{V}.$$

A random vector $\hat{\beta}$ is called an OLS estimator if $\hat{\beta}$ is a point of global minimum of $Q_{\text{ols}}(\beta)$ on a sphere $\|\beta\| = 1$, i.e. $\hat{\beta}$ is a solution to the following optimization problem:

$$\begin{cases} Q_{\text{ols}}(\beta) \to \min, \\ \|\beta\| = 1. \end{cases} \tag{5}$$

The minimum exists because $Q_{\text{ols}}(\beta)$ is a continuous function in β and the sphere is a compact set in $\mathbb{V}$.

Let

$$\psi_{\text{ols}}(x)(A, b, d) := (x^\top Ax + b^\top x + d)(xx^\top, x, 1), \quad (A, b, d) \in \mathbb{V}, \;\; x \in \mathbb{R}^n.$$

$\psi_{\text{ols}}(x)$ is a self-adjoint linear operator in $\mathbb{V}$, such that

$$q_{\text{ols}}(\beta, x) = \langle \psi_{\text{ols}}(x)\beta, \beta \rangle, \quad \beta \in \mathbb{V}, \;\; x \in \mathbb{R}^n.$$

Denote

$$\overline{\Psi}_{\text{ols}} := \sum_{l=1}^{m} \psi_{\text{ols}}(\bar{x}_l), \quad \Psi_{\text{ols}} := \sum_{l=1}^{m} \psi_{\text{ols}}(x_l).$$

Then $\overline{\Psi}_{\text{ols}}$ and Ψ_{ols} are self-adjoint operators, such that for all $\beta \in \mathbb{V}$

$$\bar{Q}_{\text{ols}}(\beta) = \langle \overline{\Psi}_{\text{ols}}\beta, \beta \rangle, \quad Q_{\text{ols}}(\beta) = \langle \Psi_{\text{ols}}\beta, \beta \rangle.$$

Note that

$$\lambda_{\min}(\overline{\Psi}_{\text{ols}}) = 0. \tag{6}$$

Next we express problem (5) in terms of Ψ_{ols}. The extremal equation implies that all solutions to (5) must be eigenvectors of the operator Ψ_{ols}. If β is an eigenvector of the operator Ψ_{ols} and $\|\beta\| = 1$, then $Q_{\text{ols}}(\beta)$ is a corresponding eigenvalue. Hence the solutions to (5) are normalized eigenvectors of Ψ_{ols}, corresponding to the smallest eigenvalue. Problem (5) is equivalent to the system

$$\begin{cases} \Psi_{\text{ols}}\beta = \lambda_{\min}(\Psi_{\text{ols}})\beta, \\ \|\beta\| = 1. \end{cases} \tag{7}$$

2.2.2. ALS estimator

The elementary score function of the ALS estimator is a solution to the following deconvolution problem:

$$\mathbb{E} q_{\text{als}}(\beta, \bar{x} + \tilde{x}) = q_{\text{ols}}(\beta, \bar{x}), \quad \tilde{x} \sim N(0, \sigma^2 I_n), \;\; \bar{x} \in \mathbb{R}^n, \;\; \beta \in \mathbb{V}. \tag{8}$$

In Appendix A we show that

$$q_{\text{als}}((A,b,d),x) := (x^\top Ax + b^\top x + d - \sigma^2 \operatorname{tr} A)^2 \\ - \sigma^2(4\|Ax\|^2 + 4b^\top Ax + \|b\|^2) + 2\sigma^4\|A\|_F^2 \tag{9}$$

is a solution to (8).

For all $\beta \in \mathbb{V}$ denote

$$Q_{\text{als}}(\beta) := \sum_{l=1}^{m} q_{\text{als}}(\beta, x_l), \quad \Psi_{\text{als}} := \sum_{l=1}^{m} \psi_{\text{als}}(x_l).$$

The linear self-adjoint operator $\psi_{\text{als}}(x)$ satisfies $\langle \psi_{\text{als}}(x)\beta, \beta\rangle = q_{\text{als}}(\beta, x)$, and Ψ_{als} is a self-adjoint linear operator, such that

$$Q_{\text{als}}(\beta) = \langle \Psi_{\text{als}}\beta, \beta\rangle, \quad \beta \in \mathbb{V}.$$

A random vector $\hat\beta$ is called an ALS1 estimator if it is a solution to the following optimization problem:

$$\begin{cases} Q_{\text{als}}(\beta) \to \min, \\ \|\beta\| = 1. \end{cases} \tag{10}$$

Similarly to the OLS estimator, such a random vector exists. Problem (10) is equivalent to the following system:

$$\begin{cases} \Psi_{\text{als}}(\beta) = \lambda_{\min}(\Psi_{\text{als}})\beta, \\ \|\beta\| = 1. \end{cases}$$

2.2.3. Translation-invariant ALS (TALS) estimator

Let

$$V_1 := \{(A,b,d) \in \mathbb{V} : \|A\|_F = 1\}.$$

We define a TALS estimator $\hat\beta$ as a random vector such that

(1) if there exists $\min_{\beta \in V_1} Q_{\text{als}}(\beta)$, then $\hat\beta$ is a minimum point (i.e., a solution to the optimization problem (11));
(2) $\hat\beta$ is arbitrary if the minimum does not exist.

The corresponding optimization problem is

$$\begin{cases} Q_{\text{als}}(A,b,d) \to \min, \\ \|A\|_F = 1. \end{cases} \tag{11}$$

Such a random vector $\hat\beta$ exists. The minimum exists if and only if Q_{als} is bounded from below on the set V_1.

2.2.4. ALS estimator with unknown variance σ^2

In the criterion function for the ALS estimator, substitute $D \in \mathbb{R}$ in place of σ^2 and denote

$$q_D((A,b,d),x) := (x^\top Ax + b^\top x + d - D \operatorname{tr} A)^2 \\ - D(4\|Ax\|^2 + 4b^\top Ax + \|b\|^2) + 2D^2\|A\|_F^2. \tag{12}$$

Let $\psi_D(x)$ be a self-adjoint operator, such that

$$\langle\psi_D(x)\beta,\beta\rangle = q_D(\beta,x), \quad \beta\in\mathbb{V}, \ \ x\in\mathbb{R}^n.$$

(The operator ψ_D is the same as in [5].)

Denote

$$Q_D(\beta) := \sum_{l=1}^m q_D(\beta,x_l), \quad \Psi_D := \sum_{l=1}^m \psi_D(x_l).$$

Setting $D=0$ or $D=\sigma^2$, we obtain the criterion function for the OLS or ALS1 estimators, respectively:

$$Q_{\mathrm{ols}}(\beta) = Q_0(\beta), \quad \Psi_{\mathrm{ols}} = \Psi_0,$$
$$Q_{\mathrm{als}}(\beta) = Q_{\sigma^2}(\beta), \quad \Psi_{\mathrm{als}} = \Psi_{\sigma^2}.$$

If $D<0$ and $x\in\mathbb{R}^n$, then the quadratic form $q_D(\beta,x)$ is positive definite. Indeed

$$q_D((A,b,d),x) = (x^\top Ax + b^\top x + d - D\,\mathrm{tr}\,A)^2 - D\|2Ax+b\|^2 + 2D^2\|A\|_F^2$$

and

$$q_D((A,b,d),x) \geqslant 2D^2\|A\|_F^2 > 0 \quad \text{if } D<0, \ \ A\neq 0,$$
$$q_D((A,b,d),x) \geqslant -D\|b\|^2 > 0 \quad \text{if } D<0, \ \ A=0, \ \ b\neq 0,$$
$$q_D((A,b,d),x) = d^2 > 0 \quad \text{if } D<0, \ \ A=0, \ \ b=0, \ \ d\neq 0.$$

Therefore, for $D<0$ the quadratic form $Q_D(\beta)$ is positive definite.

If $D=0$, then $\lambda_{\min}(\Psi_D)\geqslant 0$ [5, Lemma 6], so that $Q_0(\beta)$ is a positive semidefinite form.

Expand Ψ_D and $Q_D(\beta)$ in the powers of $D-\sigma^2$:

$$Q_D(\beta) = (D-\sigma^2)^2 Q_{\mathrm{q}}(\beta) - (D-\sigma^2)Q_{1\sigma}(\beta) + Q_{\mathrm{als}}(\beta), \tag{13}$$

where for $(A,b,d)\in\mathbb{V}$

$$Q_{\mathrm{q}}(A,b,d) := m((\mathrm{tr}A)^2 + 2\|A\|_F^2),$$
$$Q_{1\sigma}(A,b,d) := \sum_{l=1}^m q_{1\sigma}((A,b,d),x_l),$$

with

$$q_{1\sigma}((A,b,d),x) := 2(x^\top Ax + b^\top x + d - \sigma^2\,\mathrm{tr}A)\mathrm{tr}A + 4\|Ax\|^2 + 4b^\top Ax + \|b\|^2 - 4\sigma^2\|A\|_F^2. \tag{14}$$

Observe that

$$\mathbb{E}q_{1\sigma}(\beta,\bar{x}+\tilde{x}) = q_{10}(\beta,\bar{x}), \quad \tilde{x}\sim N(0,\sigma^2 I_n), \ \ \bar{x}\in\mathbb{R}^n, \ \ \beta\in\mathbb{V},$$

with

$$q_{10}((A,b,d),x) := 2(x^\top Ax + b^\top x + d)\mathrm{tr}A + 4\|Ax\|^2 + 4b^\top Ax + \|b\|^2$$

and define

$$\bar{Q}_{10}(\beta) := \mathbb{E}Q_{1\sigma}(\beta) = \sum_{l=1}^m q_{10}(\beta,\bar{x}_l), \quad \beta\in\mathbb{V}.$$

Due to the linear isomorphism between the space of quadratic forms and the space of self-adjoint linear operators, there exist self-adjoint operators Ψ_q, $\Psi_{1\sigma}$, $\overline{\Psi}_{10}$ in $\mathbb{V}$, such that for all $\beta \in \mathbb{V}$

$$Q_q(\beta) = \langle \Psi_q \beta, \beta \rangle, \quad Q_{1\sigma}(\beta) = \langle \Psi_{1\sigma}\beta, \beta\rangle, \quad \bar{Q}_{10}(\beta) = \langle \overline{\Psi}_{10}\beta, \beta\rangle,$$

$$\Psi_D = (D - \sigma^2)^2 \Psi_q - (D - \sigma^2)\Psi_{1\sigma} + \Psi_{als} \quad \text{for } D \in \mathbb{R},$$

$$\mathbb{E}\Psi_{1\sigma} = \overline{\Psi}_{10}.$$

By (6),

$$\lambda_{\min}(\mathbb{E}\Psi_{\sigma^2}) = \lambda_{\min}(\mathbb{E}\Psi_{als}) = 0,$$

so we define an estimate $\widehat{D}$ for the variance σ^2 of measurement error as a solution to the equation

$$\lambda_{\min}(\Psi_D) = 0. \tag{15}$$

Eq. (15) has no solution $D < 0$ because $\lambda_{\min}(\Psi_D) > 0$ if $D < 0$. We prove that it has a unique solution $D \geqslant 0$, see Theorem 14.

The ALS2 estimator is defined similarly to the ALS1 estimator with $Q_{als}(\beta)$ replaced by $Q_{\widehat{D}}(\beta)$. But as $\lambda_{\min}(\Psi_{\widehat{D}}) = 0$ and hence $\min_{\|\beta\|=1} Q_{\widehat{D}}(\beta) = 0$, we can simplify the definition of the ALS2 estimator.

$\hat{\beta}$ is called an ALS2 estimator if $\hat{\beta}$ is a random vector, such that

$$\begin{cases} Q_{\widehat{D}}(\hat{\beta}) = 0, \\ \|\hat{\beta}\| = 1. \end{cases} \tag{16}$$

The corresponding eigenvector problem is

$$\begin{cases} \Psi_{\widehat{D}}\beta = 0, \\ \|\beta\| = 1. \end{cases}$$

Note that $\widehat{D}$ is a random variable, because $\{\widehat{D} < D\} = \{Q_D \text{ is indefinite}\}$ is a random event, for the proof of the last equality see Corollary-remark 15.

D is a solution to (15) if and only if D satisfies the conditions

$$\begin{cases} \exists\, \beta \in \mathbb{V},\ \|\beta\| = 1 : Q_D(\beta) = 0, \\ \forall\, \beta_1 \in \mathbb{V} : Q_D(\beta_1) \geqslant 0. \end{cases}$$

A joint estimation problem for $\widehat{D}$ and $\hat{\beta}$ is

$$\begin{cases} Q_{\hat{D}}(\hat{\beta}) = 0, \\ \forall\, \beta \in \mathbb{V} : Q_{\hat{D}}(\beta) \geqslant 0, \\ \|\hat{\beta}\| = 1. \end{cases} \tag{17}$$

2.3. Conditions and their consequences

We borrow conditions (iii) and (iv) from [5].

(iii) There exist $m_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$, such that

$$\forall m \geqslant m_0 : \lambda_2\left(\frac{1}{m}\overline{\Psi}_{\mathrm{ols}}\right) \geqslant \varepsilon_0.$$

Now we show that under condition (iii) the true conic cannot be a part of a hyperplane.

Lemma 1. *Let condition* (iii) *hold. Then there is no hyperplane that contains all points* $\bar{x}_l$, $l \geqslant 1$.

Proof. Suppose that all points $\bar{x}_l$ lie on a hyperplane $b^\top x + d = 0$, $b \neq 0$. The equation of the hyperplane can be written as $x^\top bb^\top x + 2db^\top x + d^2 = 0$. Then for all m one has $\bar{Q}_{\mathrm{ols}}(0, b, d) = 0$ as well as $\bar{Q}_{\mathrm{ols}}(bb^\top, 2db, d^2) = 0$. Hence 0 is a multiple eigenvalue of $\overline{\Psi}_{\mathrm{ols}}$. This contradicts condition (iii). □

Corollary 2. *Suppose that equalities* (2) and (3), *and condition* (iii) *hold. Then* $\bar{A} \neq 0$.

The next lemma relates the sample moments of the true vectors to the norm of the matrix $\bar{A}$.

Lemma 3. *Let equalities* (2), (3), *and condition* (iii) *hold. Then for all* $m \geqslant m_0$

$$\lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix}\bar{x}_l\bar{x}_l^\top & \bar{x}_l\\ \bar{x}_l^\top & 1\end{pmatrix}\right) \geqslant \varepsilon_0\|\bar{A}\|_F^2.$$

Here m_0 *and* ε_0 *come from condition* (iii).

Proof. As $\bar{\beta}$ is a normalized eigenvector of $\overline{\Psi}_{\mathrm{ols}}$ corresponding to the eigenvalue 0, condition (iii) is equivalent to

$$\bar{Q}_{\mathrm{ols}}(\beta) \geqslant m\varepsilon_0(\|\beta\|^2 - \langle\beta, \bar{\beta}\rangle^2) \quad \text{for all } \beta \in \mathbb{V}, \quad m \geqslant m_0. \tag{18}$$

By definition of $\bar{Q}_{\mathrm{ols}}(\beta)$

$$\begin{pmatrix} b\\ d\end{pmatrix}^\top \left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix}\bar{x}_l\bar{x}_l^\top & \bar{x}_l\\ \bar{x}_l^\top & 1\end{pmatrix}\right)\begin{pmatrix} b\\ d\end{pmatrix} = \frac{1}{m}\sum_{l=1}^{m}(b^\top\bar{x}_l + d)^2 = \frac{1}{m}\bar{Q}_{\mathrm{ols}}(0, b, d). \tag{19}$$

By (2) and the Schwarz inequality

$$\begin{aligned}\|(0, b, d)\|^2 - \langle(0, b, d), \bar{\beta}\rangle^2 &= \|(0, b, d)\|^2 - \langle(0, b, d), (0, \bar{b}, \bar{d})\rangle^2\\ &\geqslant \|(0, b, d)\|^2(1 - \|(0, \bar{b}, \bar{d})\|^2)\\ &= (\|b\|^2 + d^2)(1 - \|\bar{b}\|^2 - \bar{d}^2)\\ &= \left\|\begin{pmatrix} b\\ d\end{pmatrix}\right\|^2 \|\bar{A}\|_F^2.\end{aligned} \tag{20}$$

By (18)–(20)

$$\begin{pmatrix} b \\ d \end{pmatrix}^\top \left(\frac{1}{m} \sum_{l=1}^m \begin{pmatrix} \bar{x}_l \bar{x}_l^\top & \bar{x}_l \\ \bar{x}_l^\top & 1 \end{pmatrix} \right) \begin{pmatrix} b \\ d \end{pmatrix} \geqslant \varepsilon_0 \left\| \begin{pmatrix} b \\ d \end{pmatrix} \right\|^2 \|\bar{A}\|_F^2$$

for all $m \geqslant m_0$, $b \in \mathbb{R}^n$, $d \in \mathbb{R}$. This proves the lemma. □

Corollary 4. *Suppose that equalities* (2), (3), *and condition* (iii) *hold. Let* $x_0 \in \mathbb{R}^n$. *Then for all* $m \geqslant m_0$

$$\lambda_{\min} \left(\frac{1}{m} \sum_{l=1}^m (\bar{x}_l - x_0)(\bar{x}_l - x_0)^\top \right) \geqslant \varepsilon_0 \|\bar{A}\|_F^2.$$

Proof. By Lemma 3 for all $b \in \mathbb{R}^n$, $m \geqslant m_0$,

$$\begin{aligned} b^\top \left(\frac{1}{m} \sum_{l=1}^m (\bar{x}_l - x_0)(\bar{x}_l - x_0)^\top \right) b &= \begin{pmatrix} b \\ -x_0^\top b \end{pmatrix}^\top \left(\frac{1}{m} \sum_{l=1}^m \begin{pmatrix} \bar{x}_l \bar{x}_l^\top & \bar{x}_l \\ \bar{x}_l^\top & 1 \end{pmatrix} \right) \begin{pmatrix} b \\ -x_0^\top b \end{pmatrix} \\ &\geqslant \varepsilon_0 \|\bar{A}\|_F^2 (\|b\|^2 + (b^\top x_0)^2) \geqslant \varepsilon_0 \|\bar{A}\|_F^2 \|b\|^2. \end{aligned}$$

□

Now we obtain a lower bound for a component of the limit objective function.

Lemma 5. *Let equalities* (3), (2), *and condition* (iii) *hold. Then for all* $m \geqslant m_0$

$$\bar{Q}_{10}(\bar{\beta}) \geqslant m \varepsilon_0 \|\bar{A}\|_F^4.$$

Proof. By (3)

$$\begin{aligned} \bar{Q}_{10}(\bar{\beta}) &= \sum_{l=1}^m \|2\bar{A}\bar{x}_l + \bar{b}\|^2 \\ &= \operatorname{tr} \left\{ \begin{pmatrix} 2\bar{A} \\ \bar{b}^\top \end{pmatrix}^\top \sum_{l=1}^m \begin{pmatrix} \bar{x}_l \bar{x}_l^\top & \bar{x}_l \\ \bar{x}_l^\top & 1 \end{pmatrix} \begin{pmatrix} 2\bar{A} \\ \bar{b}^\top \end{pmatrix} \right\} \\ &\geqslant (4\|\bar{A}\|_F^2 + \|\bar{b}\|^2) \lambda_{\min} \left(\sum_{l=1}^m \begin{pmatrix} \bar{x}_l \bar{x}_l^\top & \bar{x}_l \\ \bar{x}_l^\top & 1 \end{pmatrix} \right). \end{aligned}$$

The bound from Lemma 3 completes the proof. □

We combine the proofs of Lemma 4 and Corollary 5 from [5] about the convergence of the operator which represents the objective function. The following growth bound will be needed.

(iv) There exist $C_1 > 0$ and $\gamma \in [0, 1)$, such that for all $m \geqslant 1$

$$\frac{1}{m} \sum_{l=1}^m \|\bar{x}_l\|^6 \leqslant C_1 m^\gamma.$$

Lemma 6 (*Kukush et al., see [5]*). *Let conditions* (4), (i), (ii), *and* (iv) *hold. Then*

$$\left\| \tfrac{1}{m} (\Psi_{\text{als}} - \overline{\Psi}_{\text{ols}}) \right\| \to 0 \quad \textit{as } m \to \infty \ \ \textit{a.s.}$$

Moreover, the sequence

$$\left\| \tfrac{1}{m}\Psi_{\text{als}} - \tfrac{1}{m}\overline{\Psi}_{\text{ols}} \right\| m^{(1-\gamma)/2}, \quad m \geqslant 1$$

is stochastically bounded, and for any $\lambda < (1-\gamma)/2$

$$\left\| \tfrac{1}{m}\Psi_{\text{als}} - \tfrac{1}{m}\overline{\Psi}_{\text{ols}} \right\| m^{\lambda} \to 0 \quad \textit{as } m \to \infty \ \textit{a.s.}$$

One can replace condition (iv) with the following weaker one.

(iv-) $$\sum_{l=1}^{\infty} \frac{1}{l^2} \|\bar{x}_l\|^6 < \infty.$$

Indeed, condition (iv) implies (iv-), which can be proved by Abelian transformation.

Lemma 7. *Let conditions* (4), (i), (ii), *and* (iv-) *hold. Then*

$$\left\| \tfrac{1}{m}(\Psi_{\text{als}} - \overline{\Psi}_{\text{ols}}) \right\| \to 0 \quad \textit{as } m \to \infty \ \textit{a.s.} \tag{21}$$

Find proof in Appendix A.

Lemma 8. *Let conditions* (4), (i), (ii), *and* (iv-) *hold. Then*

$$\left\| \tfrac{1}{m}\Psi_{1\sigma} - \tfrac{1}{m}\overline{\Psi}_{10} \right\| \to 0 \quad \textit{as } m \to \infty \ \textit{a.s.}$$

Proof. By Lemma 7,

$$\tfrac{1}{m}(\Psi_{\text{als}} - \overline{\Psi}_{\text{ols}}) \to 0 \quad \text{as } m \to \infty \ \text{a.s.}$$

Define the linear operators

$$\pi : \mathbb{V} \to \mathbb{R}^n, \quad \pi(A, b, d) = b, \quad \text{and} \quad \pi^* : \mathbb{R}^n \to \mathbb{V}, \quad \pi^*(b) = (0, b, 0),$$

and remember the notation $\mathfrak{b}_{n_\beta} = (0, 0, 1) \in \mathbb{V}$. As

$$\sum_{i=1}^{m} (x_l x_l^\top - \sigma^2 I - \bar{x}_l \bar{x}_l^\top) = [\pi\,(\Psi_{\text{als}} - \overline{\Psi}_{\text{ols}})\,\pi^*],$$

$$\sum_{i=1}^{m} (x_l - \bar{x}_l) = \pi\,(\Psi_{\text{als}} - \overline{\Psi}_{\text{ols}})\,\mathfrak{b}_{n_\beta},$$

then

$$\frac{1}{m}\sum_{l=1}^{m} (x_l x_l^\top - \sigma^2 I - \bar{x}_l \bar{x}_l^\top) \to 0 \quad \text{as } m \to \infty \ \text{a.s.} \tag{22}$$

$$\frac{1}{m}\sum_{l=1}^{m} (x_l - \bar{x}_l) \to 0 \quad \text{as } m \to \infty \ \text{a.s.} \tag{23}$$

As

$$q_{1\sigma}((A,b,d),x) = 2(\operatorname{tr}(A(xx^\top - \sigma^2 I)) + b^\top x + d)\operatorname{tr} A + 4\operatorname{tr}(A^2(xx^\top - \sigma^2 I)) + 4b^\top Ax + \|b\|^2,$$

$$q_{10}((A,b,d),x) = 2(\operatorname{tr}(Axx^\top) + b^\top x + d)\operatorname{tr} A + 4\operatorname{tr}(A^2 xx^\top) + 4b^\top Ax + \|b\|^2,$$

one has

$$Q_{1\sigma}(A,b,d) - \bar{Q}_{10}(A,b,d) = 2\left(\operatorname{tr}\left(A\sum_{l=1}^m (x_l x_l^\top - \sigma^2 I - \bar{x}_l\bar{x}_l^\top)\right) + b^\top \sum_{l=1}^m (x_l - \bar{x}_l)\right)\operatorname{tr} A + 4\operatorname{tr}\left(A^2 \sum_{l=1}^m (x_l x_l^\top - \sigma^2 I - \bar{x}_l\bar{x}_l^\top)\right) + 4b^\top \sum_{l=1}^m (x_l - \bar{x}_l).$$

Therefore, by (22), (23), for all $\beta \in \mathbb{V}$

$$\tfrac{1}{m}(Q_{1\sigma}(\beta) - \bar{Q}_{10}(\beta)) \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

Then one can conclude convergence for the operators. □

Now we obtain a lower bound for the sample covariance matrix. It is used together with the contrast inequalities presented in Section 3.1.

Lemma 9. *Let* (3), (4), (2), *and conditions* (i)–(iii), *and* (iv-) *hold. Then almost surely*

$$\liminf_{m\to\infty} \lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^m x_l x_l^\top - \frac{1}{m^2}\sum_{l=1}^m x_l \sum_{p=1}^m x_p^\top\right) \geqslant \sigma^2 + \varepsilon_0\|\bar{A}\|^2,$$

where ε_0 *comes from condition* (iii).

Find proof in Appendix B.

We need the following condition in order to prove consistency of the ALS2 estimator.

(v) Matrix $\bar{A}$ is nonsingular.

Condition (v) means that the true conic is central, i.e. not of a parabolic type. For $n = 2$ the true conic is either an ellipse, a hyperbola, or a couple of intersecting straight lines.

Denote

$$\bar{c} := -\frac{1}{2}\bar{A}^{-1}\bar{b} \quad \text{and} \quad \bar{A}_c := \frac{1}{\bar{c}^\top \bar{A}\bar{c} - \bar{d}}\bar{A}. \tag{24}$$

Then the true conic has the equation

$$(x - \bar{c})^\top \bar{A}_c (x - \bar{c})^\top = 1.$$

The next statement relies on condition (v) and is crucial for the proof of consistency.

Lemma 10. *Let conditions* (3), (iii), *and* (v) *hold. Then there exist C_2 and $m_0 \geqslant 1$, such that*

$$\forall m \geqslant m_0 \quad \forall \beta, \quad \|\beta\| = 1 : |\bar{Q}_{10}(\beta)| \leqslant C_2 \bar{Q}_{10}(\bar{\beta}). \tag{25}$$

Here m_0 comes from condition (iii). *The constant C_2 depends on $\bar{\beta}$ and on ε_0 given in condition* (iii).

Find proof in Appendix B.

2.4. Consistency of the TALS and ALS1 estimators

In this section we use the definitions, given in Appendix C.

Denote the matrix representations

$$A := \left[\frac{1}{m}\overline{\Psi}_{\text{ols}}\right], \quad \tilde{A} := \left[\frac{1}{m}\Psi_{\text{als}}\right],$$
$$B := [\text{Pr}_{\mathbb{S}\times 0\times 0}] = \text{diag}(\underbrace{1, \ldots, 1}_{(n^2+n)/2}, \underbrace{0, \ldots, 0}_{n+1}), \tag{26}$$

where $\text{Pr}_{\mathbb{S}\times 0\times 0}$ is an operator on $\mathbb{V}$, such that $\text{Pr}_{\mathbb{S}\times 0\times 0}(A, b, d) = (A, 0, 0)$ for all $(A, b, d) \in \mathbb{V}$.

The matrices A, B are positive semidefinite, $B \leqslant I$ in the sense of Loewner order, $B = B^2$, and $\tilde{A}$ is a symmetric matrix. Under condition (iii) $A - \varepsilon_0 \text{Pr}^{\perp}_{[\bar{\beta}]}$ is positive semidefinite for $m \geqslant m_0$, with $\text{Pr}^{\perp}_{[\bar{\beta}]}$ an orthogonal projector along $[\bar{\beta}]$. If Lemma 7 holds, then $\|\tilde{A} - A\| \to 0$, as $m \to \infty$, a.s.

Now we prove that certain matrix pairs are positive definite. We have

$$\gamma_0 := \min_{\|x\|=1} \sqrt{\varepsilon_0^2 (x^\top \text{Pr}^{\perp}_{[\bar{\beta}]} x)^2 + (x^\top Bx)^2} > 0$$

as $\bar{\beta}^\top B \bar{\beta} \neq 0$,

$$\gamma(A, B) \geqslant \gamma(\varepsilon_0 \text{Pr}^{\perp}_{[\bar{\beta}]}, B) = \gamma_0 \quad \text{for } m \geqslant m_0,$$
$$\gamma(\tilde{A}, B) \geqslant \gamma(A, B) - \|\tilde{A} - A\| \geqslant \gamma_0 - \|\tilde{A} - A\| \quad \text{for } m \geqslant m_0$$

by (C.7). Hence, by Lemma 7, almost surely $\liminf_{m\to\infty} \gamma(\tilde{A}, B) \geqslant \gamma_0 > 0$.

Whenever $\gamma(A, B) > 0$ (or $\gamma(\tilde{A}, B) > 0$), the matrix A (respectively, $\tilde{A}$) has $\text{rk}B$ real finite generalized eigenvalues w.r.t. the matrix B, and has the ($n_\beta - \text{rk}B = n+1$)-dimensional eigenspace $\text{Ker}B$. Denote the finite generalized eigenvalues by $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{\text{rk}B}$ (respectively, $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_{\text{rk}B}$ for the matrix $\tilde{A}$). As $A \geqslant 0$ and $0 \leqslant B \leqslant I$, we have

$$\lambda_k \geqslant \lambda_k(A), \quad k = 1, \ldots, \text{rk}B, \tag{27}$$

where $\lambda_k(A)$ is the kth ordinary eigenvalue of the matrix A. Inequality (27) holds true because

$$\lambda_k = \min_{\dim V=k} \max_{x\in V : \|B^{1/2}x\| \leqslant 1} x^\top Ax, \quad \lambda_k(A) = \min_{\dim V=k} \max_{x\in V : \|x\|=1} x^\top Ax,$$

and $\{x\in V : \|x\|=1\} \subset \{x\in V : \|B^{1/2}x\| \leqslant 1\}$. Here the minimum is searched for in all k-dimensional subspaces of $\mathbb{R}^{n_\beta}$.

By (27), as $\lambda_1(A) = 0$ and $\lambda_2(A) \geqslant \varepsilon_0$, we have

$$\lambda_1 = 0, \quad \lambda_2 \geqslant \varepsilon_0, \quad m \geqslant m_0.$$

(As the matrix A is singular, 0 is one of its generalized eigenvalues. Hence 0 is the least one.) Therefore,

$$\chi(\lambda_k, 0) \geqslant \chi_0, \ \ k > 1, \quad \chi(\infty, 0) \geqslant \chi_0$$

with $\chi_0 = \frac{\varepsilon_0}{\sqrt{1+\varepsilon_0^2}}$.

For the chordal distance between the pairs (A, B) and $(\tilde{A}, B)$ we have

$$\rho_D := \rho_D[(A, B), (\tilde{A}, B)] \leqslant \frac{\|B\| \, \|\tilde{A} - A\|}{\gamma(A, B)\, \gamma(\tilde{A}, B)},$$

and ρ_D almost surely tends to 0, as $m \to \infty$. (ρ_D is well-defined if $\gamma(A, B) > 0$ and $\gamma(\tilde{A}, B) > 0$.)

Since A and B are positive semidefinite, the last generalized eigenvalue of the matrix A is $\lambda_1 = 0$, and the last generalized eigenvalue of the matrix $\tilde{A}$ is either $\tilde{\lambda}_1 \in \mathbb{R}$ or ∞. Here we use the enumeration from Appendix C. By [10, Theorem IV.3.2],

$$\chi(\tilde{\lambda}_1, 0) \leqslant \rho_D,$$

$$\chi(\tilde{\lambda}_k, 0) \geqslant \chi_0 - \rho_D, \quad k > 1,$$

$$\chi(\infty, 0) \geqslant \chi_0 - \rho_D$$

whenever $\|\tilde{A} - A\| \leqslant \gamma(A, B)$ and $m \geqslant m_0$. Eventually these inequalities hold.

Apply [10, Theorem IV.3.8]. Whenever $m \geqslant m_0$, $\|\tilde{A} - A\| \leqslant \gamma(A, B)$, and $\|\tilde{A} - A\| < (\chi_0 - \rho_D)\, \gamma(\tilde{A}, B)$, there exists a generalized eigenvector x_1 of the matrix $\tilde{A}$ w.r.t. to B, corresponding to eigenvalue $\tilde{\lambda}_1$, such that

$$\sin \angle(x_1, [\bar{\beta}]) \leqslant \frac{\|\tilde{A} - A\|}{\gamma(\tilde{A}, B)(\chi_0 - \rho_D)}.$$

If in addition $2\rho_D < \chi_0$, then the generalized eigenspace corresponding to the generalized eigenvalue $\tilde{\lambda}_1$ is one-dimensional.

Summarizing we have

$$\sin \angle(x_1, [\bar{\beta}]) \to 0 \quad \text{as } m \to \infty \ \text{ a.s.}$$

Whenever $\tilde{\lambda}_1 < \tilde{\lambda}_2$ and $Bx_1 \neq 0$ (i.e., eventually), x_1 is a vector in $\mathbb{R}^n$ composed of the coordinates of the TALS estimator up to a scalar multiplier,

$$[\hat{\beta}] = \pm \frac{1}{\|Bx_1\|} x_1$$

by the definition of TALS estimator. This relation proves the following consistency statement.

Theorem 11. *Let conditions* (2)–(4), *and* (i)–(iii), *and* (iv-) *hold. Let $\hat{\beta}_m$ be the TALS estimator defined in Section* 2.2 *for sample size m. Then*

$$\sin\angle(\hat{\beta}_m, \bar{\beta}) \to 0 \quad \textit{as } m \to \infty \ \textit{a.s.},$$

$$\operatorname{dist}\left(\hat{\beta}_m, \left\{\pm\frac{1}{\|\bar{A}\|_F}\bar{\beta}\right\}\right) \to 0 \quad \textit{as } m \to \infty \ \textit{a.s.}$$

The latter statement of Theorem 11 holds due to the relations

$$\operatorname{dist}\left(\frac{\hat{\beta}_m}{\|\hat{\beta}_m\|}, \pm\bar{\beta}\right) \to 0 \quad \text{as } m \to \infty \ \text{a.s.}$$

which follows from the first one by (C.2), $\hat{\beta}_m = f\left(\frac{\hat{\beta}_m}{\|\hat{\beta}_m\|}\right)$, and $\frac{1}{\|\bar{A}\|_F}\bar{\beta} = f(\bar{\beta})$, where $f(A, b, d) := \frac{1}{\|A\|_F}(A, b, d)$ is an odd continuous function.

Remark 12. If the condition (iv-) is replaced with (iv), then the rate of consistency is

$$\operatorname{dist}\left(\hat{\beta}_m, \left\{\pm\frac{1}{\|\bar{A}\|_F}\bar{\beta}\right\}\right) m^{\frac{\gamma-1}{2}} = O_p(1) \quad \text{as } m \to \infty, \tag{28}$$

and for any $\lambda < \frac{\gamma-1}{2}$

$$\operatorname{dist}\left(\hat{\beta}_m, \left\{\pm\frac{1}{\|\bar{A}\|_F}\bar{\beta}\right\}\right) m^{\lambda} \to 0 \quad \text{as } m \to \infty \ \text{a.s.} \tag{29}$$

Remark 13. Let $\hat{\beta}_{\text{als}}$ be the ALS1 estimator defined in (10). Under the conditions of Theorem 11,

$$\operatorname{dist}(\hat{\beta}_{\text{als}}, \{\pm\bar{\beta}\}) \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

The proof is easier than the proof of Theorem 11. Choosing $B = I_{n_\beta}$ instead of (26), one gets the consistency of the ALS1 estimator. We mention that this statement is proved in [5] under a slightly different condition, namely condition (iv-) is replaced by condition (iv).

3. The ALS2 estimator

In this section we deal with the case when the error variance is unknown. We prove the consistency of the estimate.

Denote by

$$S := \frac{1}{m}\sum_{l=1}^{m}\left(x_l - \frac{1}{m}\sum_{p=1}^{m}x_p\right)\left(x_l - \frac{1}{m}\sum_{q=1}^{m}x_q\right)^{\top}$$

the sample covariance matrix, and its least eigenvalue by

$$s := \lambda_{\min}(S) \geqslant 0.$$

3.1. Uniqueness of the solution to the estimating equation for the error variance

Remember the criterion function

$$Q_D(A,b,d) = \sum_{l=1}^{m}[(x_l^\top A x_l + b^\top x_l + d - D\,\mathrm{tr} A)^2 - D\|2Ax_l + b\|^2 + 2D^2\|A\|_F^2].$$

Consider the coefficient of $(-D)$:

$$\sum_{l=1}^{m}\|2Ax_l+b\|^2 = \sum_{l=1}^{m}\left\|2A\left(x_l - \frac{1}{m}\sum_{p=1}^{m}x_p\right)\right\|^2 + m\left\|2A\frac{1}{m}\sum_{p=1}^{m}x_p + b\right\|^2$$
$$= 4m\,\mathrm{tr}(ASA) + \frac{1}{m}\left\|\sum_{p=1}^{m}(2Ax_p+b)\right\|^2 \geqslant 4ms\|A\|_F^2.$$

Now we apply this bound

$$Q_{D_2}(A,b,d+D_2\,\mathrm{tr}\,A) - Q_{D_1}(A,b,d+D_1\,\mathrm{tr}\,A)$$
$$= -D_2\sum_{l=1}^{m}\|2Ax_l+b\|^2 + D_1\sum_{l=1}^{m}\|2Ax_l+b\|^2 + 2D_2^2 m\|A\|_F^2 - 2D_1^2 m\|A\|_F^2$$
$$= -(D_2-D_1)\left(\sum_{l=1}^{m}\|2Ax_l+b\|^2 - 2(D_1+D_2)m\|A\|_F^2\right).$$

If $D_1 \leqslant D_2$, then

$$Q_{D_2}(A,b,d+D_2\,\mathrm{tr}\,A) - Q_{D_1}(A,b,d+D_1\,\mathrm{tr}\,A)$$
$$\leqslant -(D_2-D_1)(4s-2D_1-2D_2)m\|A\|_F^2.$$

Moreover if $D_1 \leqslant D_2 \leqslant s$, then

$$Q_{D_2}(A,b,d+D_2\,\mathrm{tr}\,A) - Q_{D_1}(A,b,d+D_1\,\mathrm{tr}\,A) \leqslant -2(D_2-D_1)^2 m\|A\|_F^2. \tag{30}$$

Theorem 14. *The equation in D*

$$\lambda_{\min}(\Psi_D) = 0$$

has a unique solution.

Proof. If $D < 0$, then the quadratic form $Q_D(\beta)$ is positive definite, and $\lambda_{\min}(\Psi_D) > 0$. If $D = 0$, then $\lambda_{\min}(\Psi_D) \geqslant 0$, see [5, Lemma 6].

If $b \neq 0$, the expression

$$Q_D(0,b,d) = \left(\sum_{l=1}^{m}(b^\top x_l + d)^2\right) - Dm\|b\|^2 \tag{31}$$

is strictly decreasing in D. From (31) we get

$$Q_D\left(0,b,-\tfrac{1}{m}\sum_{l=1}^{m}b^\top x_l\right) = mb^\top Sb - Dm\|b\|^2.$$

If in addition b is equal to the eigenvector of S corresponding to the least eigenvalue, then

$$Q_D\left(0, b, -\frac{1}{m}\sum_{l=1}^{m} b^\top x_l\right) = m(s - D)\|b\|^2.$$

Therefore for $D > s$, the quadratic form $Q_D(\beta)$ is not positive semidefinite and $\lambda_{\min}(\Psi_D) < 0$.

Since $\lambda_{\min}(\Psi_D)$ is continuous with respect to D, there exists $\widehat{D}$, $0 \leqslant \widehat{D} \leqslant s$, such that $\lambda_{\min}(\Psi_{\widehat{D}}) = 0$.

Suppose that Eq. (15) has two solutions, $D_1 < D_2$. Then $D_2 \leqslant s$. There exists $(A, b, d) \neq 0$, such that $Q_{D_1}(A, b, d) = 0$. Since $Q_{D_1}(0, 0, d) = md^2$, we have $(A, b) \neq (0, 0)$. Then by (30) (if $A \neq 0$) or by (31) (if $A = 0$, $b \neq 0$), we have $Q_{D_2}(A, b, d + (D_2 - D_1)\operatorname{tr} A) < 0$. This contradicts the assumption $\lambda_{\min}(\Psi_{D_2}) = 0$. □

Corollary-remark 15. *There exists $\widehat{D}$, $0 \leqslant \widehat{D} \leqslant s$, such that*
- *for $D < \widehat{D}$, the quadratic form $Q_D(\beta)$ is positive definite and $\lambda_{\min}(\Psi_D) > 0$;*
- *the quadratic form $Q_{\widehat{D}}$ is positive semidefinite, but not positive definite, and $\lambda_{\min}(\Psi_{\widehat{D}}) = 0$;*
- *for $D > \widehat{D}$, the quadratic form $Q_D(\beta)$ is indefinite and $\lambda_{\min}(\Psi_D) < 0$.*

3.2. Consistency

Lemma 16. *Let conditions* (i)–(iv) *hold. Then eventually*

$$\widehat{D} - \sigma^2 < \max\left\{\frac{2Q_{\mathrm{als}}(\bar\beta)}{\bar Q_{10}(\bar\beta)}, 0\right\}. \tag{32}$$

Proof. Due to the relationship between quadratic forms and operators and since $\|\bar\beta\| = 1$, we have

$$|Q_{\mathrm{als}}(\bar\beta)| = |Q_{\mathrm{als}}(\bar\beta) - \bar Q_{\mathrm{ols}}(\bar\beta)| \leqslant \|\Psi_{\mathrm{als}} - \overline{\Psi}_{\mathrm{ols}}\|.$$

Hence by Lemma 6,

$$\frac{1}{m} Q_{\mathrm{als}}(\bar\beta) \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

Similarly by Lemma 8,

$$\frac{1}{m}(Q_{1\sigma}(\bar\beta) - \bar Q_{10}(\bar\beta)) \to 0 \quad \text{as } m \to \infty \text{ a.s.} \tag{33}$$

Next, by Corollary 2 and Lemma 5

$$\liminf_{m\to\infty} \frac{1}{m} \bar Q_{10}(\bar\beta) > 0 \quad \text{a.s.},$$

and hence by (33) the same holds true for $Q_{1\sigma}(\bar\beta)$. Note that $\frac{1}{m} Q_{\mathrm{q}}(\bar\beta)$ does not depend on m, where Q_{q} is given after (13). Hence eventually

$$4Q_{\mathrm{als}}(\bar\beta) Q_{\mathrm{q}}(\bar\beta) + (Q_{1\sigma}(\bar\beta) - \bar Q_{10}(\bar\beta))^2 < Q_{1\sigma}(\bar\beta)^2. \tag{34}$$

Then the discriminant $Q_{1\sigma}(\bar\beta)^2 - 4Q_{\mathrm{als}}(\bar\beta)Q_{\mathrm{q}}(\bar\beta)$ of the polynomial (13) in $D-\sigma^2$ with $\beta = \bar\beta$ is eventually positive, thus $Q_D(\bar\beta)$ eventually attains negative values for some D. Since by Corollary 15 $Q_D(\bar\beta) \geqslant 0$ for all $D < \widehat{D}$, we have

$$\widehat{D} - \sigma^2 \leqslant \frac{2Q_{\mathrm{als}}(\bar\beta)}{Q_{1\sigma}(\bar\beta) + \sqrt{Q_{1\sigma}(\bar\beta)^2 - 4Q_{\mathrm{als}}(\bar\beta)Q_{\mathrm{q}}(\bar\beta)}}. \tag{35}$$

Eventually, if $Q_{\mathrm{als}}(\bar\beta) \geqslant 0$, then

$$\widehat{D} - \sigma^2 \leqslant \frac{2Q_{\mathrm{als}}(\bar\beta)}{\bar{Q}_{10}(\bar\beta)},$$

otherwise $\widehat{D} - \sigma^2 < 0$. This holds true because of (34), (35), and $\bar{Q}_{10}(\bar\beta) \geqslant 0$. The lemma is proved. □

Theorem 17. *Let conditions* (i)–(v) *hold. Then the ALS2 estimator is strongly consistent*:

$$\operatorname{dist}(\hat\beta, \{\pm\bar\beta\}) \to 0 \quad \textit{as } m \to \infty \ \textit{a.s.}$$

Proof. Denote

$$\beta_2 := (\hat A, \hat b, \hat d + (\sigma^2 - \widehat{D})\operatorname{tr}\hat A).$$

Since $\hat\beta \neq 0$, we have $\beta_2 \neq 0$.

For a fixed m, assume that $\lambda_2\left(\frac{1}{m}\overline{\Psi}_{\mathrm{ols}}\right) \geqslant \varepsilon_0$ and both random events (32) and $s \geqslant \sigma^2$ occur. Consider the cases, whether the random event $\widehat{D} < \sigma^2$ occurs or not.

Case 1: *The random event* $\widehat{D} < \sigma^2$ *occurs.* By definition,

$$Q_{\widehat{D}}(\hat\beta) = 0. \tag{36}$$

By (30) with $D_2 = \sigma^2$ and $D_1 = \widehat{D}$ and $\hat d - \widehat{D}\operatorname{tr}\hat A$ in place of d and since $\widehat{D} < \sigma^2 \leqslant s$,

$$Q_{\mathrm{als}}(\beta_2) - Q_{\widehat{D}}(\hat\beta) \leqslant -2m(\sigma^2 - \widehat{D})^2\|\hat A\|_F^2. \tag{37}$$

By (C.6) and (iii),

$$m\varepsilon_0 \sin^2\angle(\beta_2, \bar\beta)\|\beta_2\|^2 \leqslant \bar{Q}_{\mathrm{ols}}(\beta_2). \tag{38}$$

Since by (C.5),

$$\sin\angle(\hat\beta, \beta_2)\|\beta_2\| \leqslant \|\hat\beta - \beta_2\| = (\sigma^2 - \widehat{D})|\operatorname{tr}\hat A|$$

and $(\operatorname{tr}\hat A)^2 \leqslant n\|\hat A\|_F^2$, we have

$$\frac{2m}{n}\sin^2\angle(\hat\beta, \beta_2)\|\beta_2\|^2 \leqslant 2m(\sigma^2 - \widehat{D})^2\|\hat A\|_F^2. \tag{39}$$

We sum up inequalities (36)–(39)

$$m(\tfrac{2}{n}\sin^2\angle(\hat\beta,\beta_2)+\varepsilon_0\sin^2\angle(\beta_2,\bar\beta))\|\beta_2\|^2\leqslant\|\Psi_{\mathrm{als}}-\overline{\Psi}_{\mathrm{ols}}\|\,\|\beta_2\|^2.$$

Since $\beta_2\neq 0$, we can divide both sides by $\|\beta_2\|^2$. By the Schwarz inequality and Lemma 36

$$\begin{aligned}\sin^2\angle(\hat\beta,\bar\beta)&\leqslant(\sin\angle(\hat\beta,\beta_2)+\sin\angle(\beta_2,\bar\beta))^2\\&\leqslant\left(\frac{n}{2}+\frac{1}{\varepsilon_0}\right)\left(\frac{2}{n}\sin^2\angle(\hat\beta,\beta_2)+\varepsilon_0\sin^2\angle(\beta_2,\bar\beta)\right)\\&\leqslant\left(\frac{n}{2}+\frac{1}{\varepsilon_0}\right)\frac{1}{m}\|\Psi_{\mathrm{als}}-\overline{\Psi}_{\mathrm{ols}}\|.\end{aligned}$$

Case 2: *The alternative event $\widehat{D}\geqslant\sigma^2$ occurs*. From $Q_{\widehat{D}}(\hat\beta)=0$ and (13) we have

$$(\widehat{D}-\sigma^2)^2Q_{\mathrm{q}}(\hat\beta)-(\widehat{D}-\sigma^2)Q_{1\sigma}(\hat\beta)+Q_{\mathrm{als}}(\hat\beta)=0.$$

It is clear that

$$-(\widehat{D}-\sigma^2)^2Q_{\mathrm{q}}(\hat\beta)\leqslant 0.$$

By (C.6) and (iii),

$$m\varepsilon_0\sin^2\angle(\hat\beta,\bar\beta)\leqslant\bar{Q}_{\mathrm{ols}}(\hat\beta).$$

By (32),

$$(\widehat{D}-\sigma^2)\bar{Q}_{10}(\hat\beta)\leqslant\frac{2Q_{\mathrm{als}}(\bar\beta)}{\bar{Q}_{10}(\bar\beta)}|\bar{Q}_{10}(\hat\beta)|.$$

From (32) and the relation $\|\hat\beta\|=1$, we have

$$\begin{aligned}&(\widehat{D}-\sigma^2)(Q_{1\sigma}(\hat\beta)-\bar{Q}_{10}(\hat\beta))\leqslant\frac{2Q_{\mathrm{als}}(\bar\beta)}{\bar{Q}_{10}(\bar\beta)}\|\Psi_{1\sigma}-\overline{\Psi}_{10}\|,\\&\bar{Q}_{\mathrm{ols}}(\hat\beta)-Q_{\mathrm{als}}(\hat\beta)\leqslant\|\Psi_{\mathrm{als}}-\overline{\Psi}_{\mathrm{ols}}\|.\end{aligned}$$

We sum these inequalities up

$$m\varepsilon_0\ \sin^2\angle(\hat\beta,\bar\beta)\leqslant\|\Psi_{\mathrm{als}}-\overline{\Psi}_{\mathrm{ols}}\|+\frac{2Q_{\mathrm{als}}(\bar\beta)}{\bar{Q}_{10}(\bar\beta)}(\|\Psi_{1\sigma}-\overline{\Psi}_{10}\|+|\bar{Q}_{10}(\hat\beta)|).$$

Since $|Q_{\mathrm{als}}(\bar\beta)|\leqslant\|\Psi_{\mathrm{als}}-\overline{\Psi}_{\mathrm{ols}}\|$,

$$m\varepsilon_0\ \sin^2\angle(\hat\beta,\bar\beta)\leqslant\|\Psi_{\mathrm{als}}-\overline{\Psi}_{\mathrm{ols}}\|\left(1+\frac{2}{\bar{Q}_{10}(\bar\beta)}\|\Psi_{1\sigma}-\overline{\Psi}_{10}\|+\frac{|\bar{Q}_{10}(\hat\beta)|}{\bar{Q}_{10}(\bar\beta)}\right).$$

By Lemmas 9 and 16, both the random events $s \geqslant \sigma^2$ and (32) hold eventually, and by condition (iii), $\lambda_2\left(\frac{1}{m}\overline{\Psi}_{\mathrm{ols}}\right) \geqslant \varepsilon_0$ holds for m large enough. Then eventually

$$\sin^2 \angle(\hat{\beta}, \bar{\beta}) \leqslant \frac{\|\Psi_{\mathrm{als}} - \overline{\Psi}_{\mathrm{ols}}\|}{m} \max\left\{\left(\frac{n}{2} + \frac{1}{\varepsilon_0}\right), \right.$$
$$\left. \frac{1}{\varepsilon_0}\left(1 + \frac{2}{\bar{Q}_{10}(\bar{\beta})}\|\Psi_{1\sigma} - \overline{\Psi}_{10}\| + \frac{|\bar{Q}_{10}(\hat{\beta})|}{\bar{Q}_{10}(\bar{\beta})}\right)\right\}. \tag{40}$$

By Lemmas 5, 6, 8, and 10,

$$\sin^2 \angle(\hat{\beta}, \bar{\beta}) \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

Due to (C.2), the consistency is proved. □

Remark 18. Condition (v) can be replaced by the following one

(vi) $\exists C_3 > 0 \, \forall m \geqslant 1 \; : \; \frac{1}{m}\sum_{l=1}^{m} \|\bar{x}_l\|^2 \leqslant C_3.$

The condition (v) was used in Lemma 10 to prove that the sequence $\{\frac{\bar{Q}_{10}(\hat{\beta})}{\bar{Q}_{10}(\bar{\beta})}, m \geqslant m_0\}$ is bounded. Under (vi) the sequence $\{\frac{1}{m}\bar{Q}_{10}(\hat{\beta}),\ m \geqslant 1\}$ is bounded. With Lemma 5, we get the desired bound.

Remark 19. In [1] a polynomial functional measurement error model is considered

$$y_i = \sum_{j=0}^{p} \beta_j \bar{x}_i^j + \tilde{y}_i, \quad x_i = \bar{x}_i + \tilde{x}_i.$$

Here $\{\tilde{y}_i\}$ and $\{\tilde{x}_i\}$ are two i.i.d. error sequences, independent of each other. For the case of known ratio $\lambda := \mathrm{var}(\tilde{y}_i)/\mathrm{var}(\tilde{x}_i)$, while the variances themselves are unknown, a certain estimation procedure for $\beta_0, \ldots, \beta_p$ is proposed. It is not clear whether that procedure is consistent or not. And in case $p = 2$ of the quadratic model, under known λ and normal errors, there is a clear way to estimate the regression coefficients consistently: just imbed the explicit quadratic model into an implicit one

$$\bar{y}_i - \sum_{j=0}^{2} \beta_j \bar{x}_i^j = 0, \quad x_i = \bar{x}_i + \tilde{x}_i, \quad y_i = \bar{y}_i + \tilde{y}_i.$$

By remark 18, under rather mild conditions the ALS2 estimator of $\beta := (\beta_0, \beta_1, \beta_2)^\top$ is consistent.

Remark 20. Under the conditions of Theorem 17, but without (v), the following holds:

$$\mathrm{dist}(\hat{\beta}, \{\pm\bar{\beta}\})\, \mathbb{I}\{\widehat{D} < \sigma^2\} \to 0 \quad \text{as } m \to \infty \text{ a.s.},$$

where $\mathbb{I}(P)$ is an indicator function of random event P.

Theorem 21 (*Consistency of the estimator of error variance*). *Let conditions* (i)–(iv) *hold. Then $\widehat{D}$ is a strictly consistent estimate of σ^2.*

Proof. Whenever the random event $\widehat{D} \leqslant \sigma^2 \leqslant s$ occurs, we sum up lines (36), (37), and the inequality

$$0 \leqslant \bar{Q}_{\mathrm{ols}}(\beta_2).$$

We get

$$2m(\sigma^2 - \widehat{D})^2 \|\hat{A}\|_F^2 \leqslant \|\Psi_{\mathrm{als}} - \overline{\Psi}_{\mathrm{ols}}\| \, \|\beta_2\|^2.$$

If $\widehat{D} \leqslant \sigma^2$, then $\|\beta_2\| \leqslant 1 + \sigma^2\sqrt{n}$. By Remark 20,

$$(\|\hat{A}\|_F - \|\bar{A}\|_F)\, \mathbb{I}\{\widehat{D} < \sigma^2\} \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

By Lemma 9, the random event $s \geqslant \sigma^2$ occurs eventually. Hence, by Corollary 2 and Lemma 6,

$$(\widehat{D} - \sigma^2)\, \mathbb{I}\{\widehat{D} < \sigma^2\} \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

Next, the convergence

$$(\widehat{D} - \sigma^2)\, \mathbb{I}\{\widehat{D} \geqslant \sigma^2\} \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

holds true due to Lemmas 6, 8, and 10.

Finally,

$$\widehat{D} - \sigma^2 \to 0 \quad \text{as } m \to \infty \text{ a.s.} \quad \square$$

4. Structural model

We considered a functional measurement error model. Now we study a structural model with random vectors $\bar{x}_l$. In this section assume that (2)–(4) hold. Introduce the following conditions:

(S1) The random vectors $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots; \tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots$ are totally independent.
(S2) The random vectors $\bar{x}_1, \bar{x}_2, \ldots$ are identically distributed.
(S3) $\tilde{x}_l$ has normal distribution, $\tilde{x}_l \sim N(0, \sigma^2 I)$, $\sigma > 0$, $l \geqslant 1$.
(S4) $\mathbb{E}\|\bar{x}_1\|^4 < \infty$.
(S5) $\lambda_2(\mathbb{E}\psi_{\mathrm{ols}}(\bar{x}_1)) > 0$.

We mention that (S4) provides the existence of $\mathbb{E}\psi_{\mathrm{ols}}(\overline{x}_1)$.

Proposition 22. *Let conditions* (S1),(S3),(S4) and (S5) *hold. Then ALS1, TALS, and ALS2 estimators are strongly consistent, i.e., the statements of Theorems* 11, 17, 21, *and Remark* 13 *hold true.*

Proof. By condition (S4) $\mathbb{E}\|x_1\|^4 < \infty$, $\mathbb{E}\|\bar{x}_1\|^2 < \infty$, and the expectations of $\psi_{\mathrm{ols}}(\bar{x}_1)$ and $\psi_{\mathrm{als}}(x_1)$ exist and are finite. By the strong law of large numbers,

$$\frac{1}{m}\overline{\Psi}_{\mathrm{ols}} \to \mathbb{E}\psi_{\mathrm{ols}}(\bar{x}_1) \quad \text{as } m \to \infty \text{ a.s.}$$

$$\frac{1}{m}\Psi_{\mathrm{als}} \to \mathbb{E}\psi_{\mathrm{als}}(x_1) = \mathbb{E}\psi_{\mathrm{ols}}(\bar{x}_1) \quad \text{as } m \to \infty \text{ a.s.}$$

Then (21) holds true. By (21) and condition (S5), there exists ε_0 such that eventually $\lambda_2\left(\frac{1}{m}\Psi_{\mathrm{ols}}\right) \geqslant \varepsilon_0$, i.e., condition (iii) holds true a.s.

The consistency of the ALS and TALS estimators follows from condition (iii) and convergence (21) similarly to the proof of Theorem 11.

As $\mathbb{E}\|\bar{x}_1\|^2 < \infty$, by the strong law of large numbers

$$\frac{1}{m}\sum_{l=1}^{m}\|\bar{x}_l\|^2 \to \mathbb{E}\|\bar{x}_1\|^2 \quad \text{as } m \to \infty \quad \text{a.s.},$$

and then condition (vi) holds true a.s. The ALS2 estimator is consistent. The proof is similar to the proof of Theorem 17 under condition (vi) (see Remark 18). □

The condition (S4) can be relaxed. Consider assumptions

(S4-) $\mathbb{E}\|\bar{x}_1\|^3 < \infty$.

(S5-) The identity "$\psi_{\text{ols}}(\bar{x}_1)\beta = 0$ a.s." implies "$\beta = k\bar{\beta}$ for some $k\in\mathbb{R}$".

Condition (S5-) means that the distribution of $\bar{x}_1$ is not concentrated on an intersection of two different conics. Under (S4) conditions (S5) and (S5-) are equivalent.

Proposition 23. *Let conditions* (S1)–(S3),(S4-), and (S5-) *hold true. Then the ALS*, *TALS*, *and ALS2 estimators are strongly consistent.*

Sketch of Proof. Condition (S4-) implies (iv) a.s. by Kolmogorov theorem about three series (see [9]). By (S4-) condition (vi-) holds true a.s. Condition (S5-) implies (iii) a.s. (with random C_3). Thus, conditions of the consistency theorems hold true a.s. with given $\bar{x}_l$, $l = 1, 2, \ldots$. □

5. Invariance of the estimates

5.1. Notations

Let the sample size m be fixed. Consider an arbitrary estimator of $\bar{\beta}$, i.e., a measurable mapping from the sample space $\mathbb{R}^{n\times m}$ into the parameter space $\mathbb{V}$. There is a natural one-to-one correspondence between $\mathbb{V}$ and the space of polynomials in n variables of degree $\leqslant 2$, namely the polynomial $x^\top Ax + b^\top x + d$ in the coordinates of x corresponds to the triple (A, b, d).

Suppose for a sample $X = [x_1, x_2, \ldots, x_m]$ that the estimate is equal to $\hat{\beta} = (\hat{A}, \hat{b}, \hat{d})$. Denote by

$$\hat{\Phi}_X(x) := x^\top \hat{A}x + \hat{b}^\top x + \hat{d}$$

the link function of the estimated conic, further referred to as *estimate of the link function*. The equation of the estimated conic is $\hat{\Phi}_X(x) = 0$.

Unfortunately problems (5), (10), (11), and (17), which are referred below as *estimation problems*, may have multiple solutions. Again fix the sample size m. If a sample X is observed, let $\widehat{\mathrm{B}}(X) \subset \mathbb{V}$ be the set of solutions to the estimation problem. Denote by

$$\mathrm{Sol}(X) := \{x \mapsto x^\top Ax + b^\top x + d \mid (A, b, d) \in \widehat{\mathrm{B}}(X)\}$$

the set of the link functions defined by the solutions to the estimation problem.

Denote the density of an n-variate normal distribution $N(0, \Sigma)$ by p_Σ. If the distribution is homogeneous, the notation $p_{\sigma^2} = p_{\sigma^2 I}$ is used.

Let $f(x)$ be a polynomial. The convolution with the normal distribution density is denoted by

$$f * p_\Sigma(x) = \mathbb{E} f(x + \tilde{x}), \quad \tilde{x} \sim N(0, \Sigma).$$

The deconvolution denoted by $f * p_{-\Sigma}$ is a polynomial such that $(f * p_{-\Sigma}) * p_\Sigma = f$. This means that for $g = f * p_{-\Sigma}$ the equality $\mathbb{E} g(x + \tilde{x}) = f(x)$ holds.

Introduce an abstract notation for functions. The composition of the functions f and T is denoted by $f \circ T$, i.e., $f \circ T(x) = f(T(x))$. The notation Φ^2 means $x \mapsto \Phi(x)^2$, and $(\Phi \circ T)^2(x) = \Phi^2 \circ T(x) = \Phi(T(x))^2$. If T is a one-to-one transformation, the inverse transformation is denoted by T^{-1}.

Let T be an affine transformation on $\mathbb{R}^n$, $T(x) = Kx + h$, where K is an $n \times n$ matrix, and let f be a polynomial. The formulae of convolution and deconvolution of the composition $f(T(x))$ are given next.

$$(f \circ T) * p_\Sigma(x) = \mathbb{E} f(T(x + \tilde{x})) = \mathbb{E} f(T(x) + K\tilde{x}) = f * p_{K\Sigma K^\top}(T(x))$$

with $\tilde{x} \sim N(0, \Sigma)$ and $K\tilde{x} \sim N(0, K\Sigma K^\top)$. The formula for the deconvolution is

$$(f \circ T) * p_{-\Sigma} = (f * p_{-K\Sigma K^\top}) \circ T, \tag{41}$$

because

$$((f * p_{-K\Sigma K^\top}) \circ T) * p_\Sigma = (f * p_{-K\Sigma K^\top} * p_{K\Sigma K^\top}) \circ T = f \circ T.$$

Let $\beta = (A, b, d) \in \mathbb{V}$ and $\Phi(x) = x^\top A x + b^\top x + d$. Let a sample $X = [x_1, x_2, \ldots, x_m]$ be fixed. As $q_{\mathrm{ols}}(\beta, x) = \Phi(x)^2$,

$$Q_{\mathrm{ols}}(\beta) = \sum_{l=1}^{m} \Phi(x_l)^2.$$

By (8), $q_{\mathrm{als}}(\beta, x) = q_{\mathrm{ols}}(\beta, x) * p_{-\sigma^2} = \Phi^2 * p_{-\sigma^2}(x)$. Hence

$$Q_{\mathrm{als}}(\beta) = \sum_{l=1}^{m} \Phi^2 * p_{-\sigma^2}(x_l).$$

Denote

$$v_1(\Phi) := \|A\|_F,$$

$$v_2(\Phi) := \sqrt{\|A\|_F^2 + \|b\|^2 + d^2}.$$

The estimation problems (5), (10), (11), and (17) are reformulated in terms of the estimators of link functions. In the following formulae, $\Phi(x)$ is a polynomial of order less than or equal to 2.

Finally, for the OLS estimator, $\Phi \in \mathrm{Sol}(X)$ if and only if Φ delivers a constrained minimum to the problem

$$\begin{cases} \sum_{l=1}^{m} \Phi(x_l)^2 \to \min, \\ v_2(\Phi) = 1. \end{cases}$$

For the ALS1 estimator, $\Phi \in \mathrm{Sol}(X)$ if and only if Φ is a solution to the problem

$$\begin{cases} \sum_{l=1}^{m} \Phi^2 * p_{-\sigma^2}(x_l) \to \min, \\ v_2(\Phi) = 1. \end{cases} \tag{42}$$

For the TALS estimator, $\Phi \in \mathrm{Sol}(X)$ if and only if Φ is a solution to the problem

$$\begin{cases} \sum_{l=1}^{m} \Phi^2 * p_{-\sigma^2}(x_l) \to \min, \\ v_1(\Phi) = 1. \end{cases} \tag{43}$$

For the ALS2 estimator, $\Phi \in \mathrm{Sol}(X)$ if and only if there exists $D \geqslant 0$, such that

$$\begin{cases} \sum_{l=1}^{m} \Phi^2 * p_{-D}(x_l) = 0, \\ \sum_{l=1}^{m} \Phi_1^2 * p_{-D}(x_l) \geqslant 0 \quad \text{for any polynomial } \Phi_1 \text{ of order } \leqslant 2, \\ v_2(\Phi) = 1. \end{cases} \tag{44}$$

5.2. *Definition of invariance*

There are infinitely many coordinate systems of an affine space. The question we consider next is how the estimated conic depends on the choice of the coordinate system.

Let in an n-dimensional affine space two coordinate systems be fixed. The transformation function is T: if a point has coordinates x in the first system, it has coordinates $y = T(x)$ in the second one. Note that $T(x)$ is of the form $T(x) = Kx + h$ with a nonsingular $n \times n$ matrix K and a vector $h \in \mathbb{R}^n$.

Let a sample on the space be given. Denote by $X := [x_1, x_2, \ldots, x_m]$ and $Y := [y_1, y_2, \ldots, y_m]$ the m-ples of the points of the sample expressed in x- and y-coordinates, respectively. The relation

$$T(x_l) = y_l, \quad l = 1, 2, \ldots, m$$

is denoted by

$$T(X) = Y.$$

Let $\hat{\Phi}_X(x)$ be an estimator of a link function. When the first coordinate system is used, the equation of the estimated conic is

$$\hat{\Phi}_X(x) = 0.$$

When the second system is used, the equation is

$$\hat{\Phi}_Y(y) = 0.$$

These equations define the same conic if and only if

$$\hat{\Phi}_X(x) = 0 \ \Leftrightarrow\ \hat{\Phi}_Y(T(x)) = 0.$$

Definition 24. Let $\hat{\Phi}$ be an estimator of a link function. Let T be an affine transformation of $\mathbb{R}^n$. The underlying estimator is called T-invariant if the following equations are equivalent:

$$\hat{\Phi}_X(x) = 0 \quad \Leftrightarrow \quad \hat{\Phi}_{T(X)}(T(x)) = 0.$$

Now we suppose that an estimation problem arises, and the estimator is not necessarily unique. If the first coordinate system is used, then the set of estimated conics is

$$S_1 := \{\{x|\Phi(x) = 0\} : \Phi \in \mathrm{Sol}(X)\}.$$

If the second coordinate system is used, then the set of estimated conics is

$$\{\{y|\Phi(y) = 0\} : \Phi \in \mathrm{Sol}(Y)\}.$$

We perform a coordinate transformation. If the second system is used for estimation procedure and the equations of the estimated conics are rewritten in x-coordinates, then the set of estimated conics is

$$S_{21} := \{\{x|\Phi(T(x)) = 0\} : \Phi \in \mathrm{Sol}(Y)\}.$$

The two sets of estimated conics are the same if and only if $S_1 = S_{21}$.

Definition 25. Fix a sample X. Consider an estimation problem. Let $T(x)$ be an affine transformation of $\mathbb{R}^n$. The problem is called $T\Rightarrow$invariant if $\forall\Phi_1 \in \mathrm{Sol}(X)\ \exists\Phi_2 \in \mathrm{Sol}(T(X))$, such that

$$\Phi_1(x) = 0 \quad \Leftrightarrow \quad \Phi_2(T(x)) = 0.$$

The problem is called $T\Leftarrow$invariant if $\forall\Phi_2 \in \mathrm{Sol}(T(X))\exists\Phi_1 \in \mathrm{Sol}(X)$, such that

$$\Phi_1(x) = 0 \quad \Leftrightarrow \quad \Phi_2(T(x)) = 0.$$

The problem is called T-invariant if it is both $T\Rightarrow$invariant and $T\Leftarrow$invariant.

Remark 26. Suppose that for any sample X, an estimation problem is $T\Rightarrow$invariant and $T^{-1}\Rightarrow$ invariant. Then for any sample X it is T-invariant. The reason is that the $T\Leftarrow$invariance for a sample X coincides with the $T^{-1}\Rightarrow$invariance for the sample $T(X)$.

The next statement concerns the relation between the invariance of an estimator and the invariance of an estimation problem.

Proposition 27. *Let an estimation problem and an estimator be given. Suppose that for any sample, whenever the estimation problem has solutions, the estimator provides one of them. Let T be an affine transformation. For a given sample X, suppose that the problem is T-invariant and its solutions define a unique conic. Then the estimator is T-invariant for the sample X.*

Proof. Let $\mathrm{Sol}(X)$ be the set of all the link functions defined by the solutions to the problem, and $\hat{\Phi}_X$ be the estimator of the link functioncorresponding to the estimator (of β). The relation between the estimation problem and the estimator is such that for any sample X', either

$$\hat{\Phi}_{X'} \in \mathrm{Sol}(X') \quad \text{or} \quad \mathrm{Sol}(X') = \varnothing.$$

The uniqueness of the estimated conic means that

$$\mathrm{Sol}(X) \neq \emptyset$$

and

$$\text{if } \Phi_1 \in \mathrm{Sol}(X) \text{ and } \Phi_2 \in \mathrm{Sol}(X) \quad \text{then} \quad \Phi_1(x) = 0 \quad \Leftrightarrow \quad \Phi_2(x) = 0. \tag{45}$$

We have $\Phi_X \in \mathrm{Sol}(X)$. Then by the $T \Rightarrow$invariance, $\mathrm{Sol}(T(X)) \neq \emptyset$, and therefore $\Phi_{(T(X))} \in \mathrm{Sol}(T(x))$. Because of the $T \Leftarrow$invariance, there exists $\Phi_1 \in \mathrm{Sol}(X)$, such that

$$\Phi_1(x) = 0 \quad \Leftrightarrow \quad \Phi_{T(X)}(T(x)) = 0. \tag{46}$$

By the relations (45) for $\Phi_2 = \Phi_X$, and by (46), the estimator is T-invariant. □

5.3. *Rotation invariance of the ALS1 estimator*

Consider the transformation $T(x) = Sx$ with an orthogonal $n \times n$ matrix S.

Theorem 28. *For any sample X, problem* (10) *is T-invariant for $T(x) = Sx$.*

Proof. Hereafter Φ is a polynomial of order $\leqslant 2$. By (41),

$$\sum_{l=1}^{m} (\Phi \circ T)^2 * p_{-\sigma^2}(x_l) = \sum_{l=1}^{m} (\Phi^2 \circ T) * p_{-\sigma^2}(x_l) = \sum_{l=1}^{m} \Phi^2 * p_{-\sigma^2}(T(x_l)), \tag{47}$$

because $S(-\sigma^2 I)S^\top = -\sigma^2 I$. We show that

$$v_2(\Phi \circ T) = v_2(\Phi).$$

Let $\Phi(x) = x^\top Ax + b^\top x + d$, $(A, b, d) \in \mathbb{V}$. Since $\|S^\top AS\|_F = \|A\|_F$, $\|S^\top b\| = \|b\|$, and $\Phi(T(x)) = x^\top S^\top ASx + (S^\top b)^\top x + d$, we have

$$v_2^2(\Phi \circ T) = \|S^\top AS\|_F^2 + \|S^\top b\|^2 + d^2 = \|A\|_F^2 + \|b\|^2 + d^2 = v_2^2(\Phi).$$

By (42), $\Phi \in \mathrm{Sol}(T(X))$ if and only if Φ is a solution to the problem

$$\begin{cases} \sum\limits_{l=1}^{m} \Phi^2 * p_{-\sigma^2}(T(x_l)) \to \min\,, \\ v_2(\Phi) = 1. \end{cases}$$

This problem is equivalent to

$$\begin{cases} \sum\limits_{l=1}^{m} (\Phi \circ T)^2 * p_{-\sigma^2}(x_l) \to \min\,, \\ v_2(\Phi \circ T) = 1. \end{cases} \tag{48}$$

Now, we prove the $T \Rightarrow$invariance. Suppose that $\Phi_1 \in \mathrm{Sol}(X)$, i.e., Φ_1 is a solution to problem (42). Then $\Phi_1 \circ T^{-1}$ is a solution to problem (48), i.e., $\Phi_1 \circ T^{-1} \in \mathrm{Sol}(T(X))$. The relation

$$\Phi_1(x) = 0 \quad \Leftrightarrow \quad \Phi_1(T^{-1}(T(x))) = 0 \tag{49}$$

is obvious. The $T\Rightarrow$invariance is proven. Since the inverse transformation $T^{-1}(y) = S^\top y$ is of the same form, the problem (10) is $T^{-1}\Rightarrow$invariant for any sample. By Remark 26, it is T-invariant. □

The same invariance holds for the estimation problem (5) for the OLS estimator.

5.4. Isometry invariance of the TALS estimator

Consider the transformation $T(x) = Sx + h$ with an orthogonal matrix S.

Theorem 29. *Problem* (11) *is T-invariant for* $T(x) = Sx + h$ *for any sample X.*

Proof. By (41), the formula (47) holds true. Next we prove that

$$v_1(\Phi \circ T) = v_1(\Phi). \tag{50}$$

Let $\Phi(x) = x^\top Ax + b^\top x + d$, $(A, b, d) \in \mathbb{V}$. Then

$$\Phi(T(x)) = x^\top S^\top ASx + (2Ah + b)^\top Sx + h^\top Ah + b^\top h + d.$$

Hence $v_1(\Phi \circ T) = \|S^\top AS\|_F = \|A\|_F = v_1(\Phi)$.

By (43), $\Phi \in \mathrm{Sol}(T(X))$ if and only if Φ is a solution to the problem

$$\begin{cases} \sum_{l=1}^{m} \Phi^2 * p_{-\sigma^2}(T(x_l)) \to \min, \\ v_1(\Phi) = 1. \end{cases}$$

By (47) and (50), this problem is equivalent to

$$\begin{cases} \sum_{l=1}^{m} (\Phi \circ T)^2 * p_{-\sigma^2}(x_l) \to \min, \\ v_1(\Phi \circ T) = 1. \end{cases} \tag{51}$$

We prove the $T\Rightarrow$invariance. Let $\Phi_1 \in \mathrm{Sol}\,X$. Then Φ_1 is a solution to (43), $\Phi_1 \circ T^{-1}$ is a solution to (51), i.e., $\Phi_1 \circ T^{-1} \in \mathrm{Sol}\,T(X)$. The equivalence (49) completes the proof of the $T\Rightarrow$invariance.

The same holds for the transformation $T^{-1}(y) = S^\top y - S^\top h$. By Remark 26, problem (11) is T-invariant. □

5.5. Similarity invariance of the ALS2 estimator

Theorem 30. *Let the transformation* $T(x)$ *be of the form* $T(x) = kSx + h$, *with an orthogonal matrix S and real* $k \neq 0$. *Then problem* (17) *is T-invariant for any sample X.*

Proof. By (41), for any real $D \geqslant 0$ we have

$$(\Phi \circ T)^2 * p_{-D} = (\Phi^2 \circ T) * p_{-D} = (\Phi^2 * p_{-k^2 D}) \circ T, \tag{52}$$

because $kS(-DI)(kS)^\top = -k^2DI$. We apply (52) to $\Phi = \Phi_1 * T^{-1}$:

$$\Phi_1^2 * p_{-D}(x) = (\Phi_1 \circ T^{-1})^2 * p_{-k^2D}(T(x)). \tag{53}$$

Now, we prove the $T\Rightarrow$invariance. Let $\Phi_1 \in \mathrm{Sol}(X)$. Then there exists $D \geqslant 0$, such that Φ satisfies (44). By (53), the first equation of (44) implies that

$$\sum_{l=1}^{m} (\Phi_1 \circ T^{-1})^2 * p_{-k^2D}(T(x_l)) = 0. \tag{54}$$

For any polynomial $\Phi_2(x)$ of order $\leqslant 2$, $\Phi_2(T(x))$ is also a polynomial of order $\leqslant 2$. Then by the second line of (44) and by (52),

$$\sum_{l=1}^{m} \Phi_2^2 * p_{-k^2D}(T(x_l)) \geqslant 0. \tag{55}$$

By the third line of (44), the polynomial Φ_1 is not identically 0. Neither is $\Phi_1 \circ T^{-1}$, thus $v_2(\Phi_1 \circ T^{-1}) \neq 0$. Denote

$$\Phi_3(x) := \frac{\Phi_1(T^{-1}(x))}{v_2(\Phi_1 \circ T^{-1})}.$$

By (54), (55), and since v_2 is homogeneous, we have

$$\begin{cases} \sum_{l=1}^{m} \Phi_3^2 * p_{-k^2D}(T(x_l)) = 0, \\ \sum_{l=1}^{m} \Phi_2^2 * p_{-k^2D}(T(x_l)) \geqslant 0 \text{ for any polynomial } \Phi_2 \text{ of order } \leqslant 2, \\ v_2(\Phi_3) = 1. \end{cases}$$

We see that the link function Φ_3 satisfies conditions (44) for the sample $T(X)$. Hence $\Phi_3 \in \mathrm{Sol}(T(X))$. Since $\Phi_3(T(x)) = \frac{1}{v_2(\Phi_1 \circ T^{-1})}\Phi_1(x)$, the polynomial $\Phi_3(T(x))$ has the same zeros as $\Phi_1(x)$. The $T\Rightarrow$invariance is proved.

The same holds true for the transformation $T^{-1}(y) = k^{-1}S^\top y - k^{-1}S^\top h$ and any sample X. By Remark 26, problem (17) is T-invariant. □

A simulation study confirming the invariance of the ALS2 estimator is given in [7].

Denote by $D(X)$ the solution to Eq. (15) with the sample X observed. (The solution is unique, see Theorem 14.) Next we show that the variance estimator is invariant under isometries.

Theorem 31. *Let the transformation T be of the form* $T(x) = Sx + h$, *with an orthogonal matrix S. Then for any sample X*,

$$D(X) = D(T(X)).$$

Proof. D is a solution to (15) if and only if there exists a polynomial Φ of order $\leqslant 2$, such that conditions (44) hold. For any sample X there exists Φ_1, such that conditions (44) hold true for $D = D(X)$, $\Phi = \Phi_1$. Then conditions (44) are satisfied for $D = D(X)$, $\Phi = \frac{1}{v_2(\Phi_1 \circ T^{-1})}\Phi_1 \circ T^{-1}$, and the sample $T(X)$. Therefore $D(X) = D(T(X))$. □

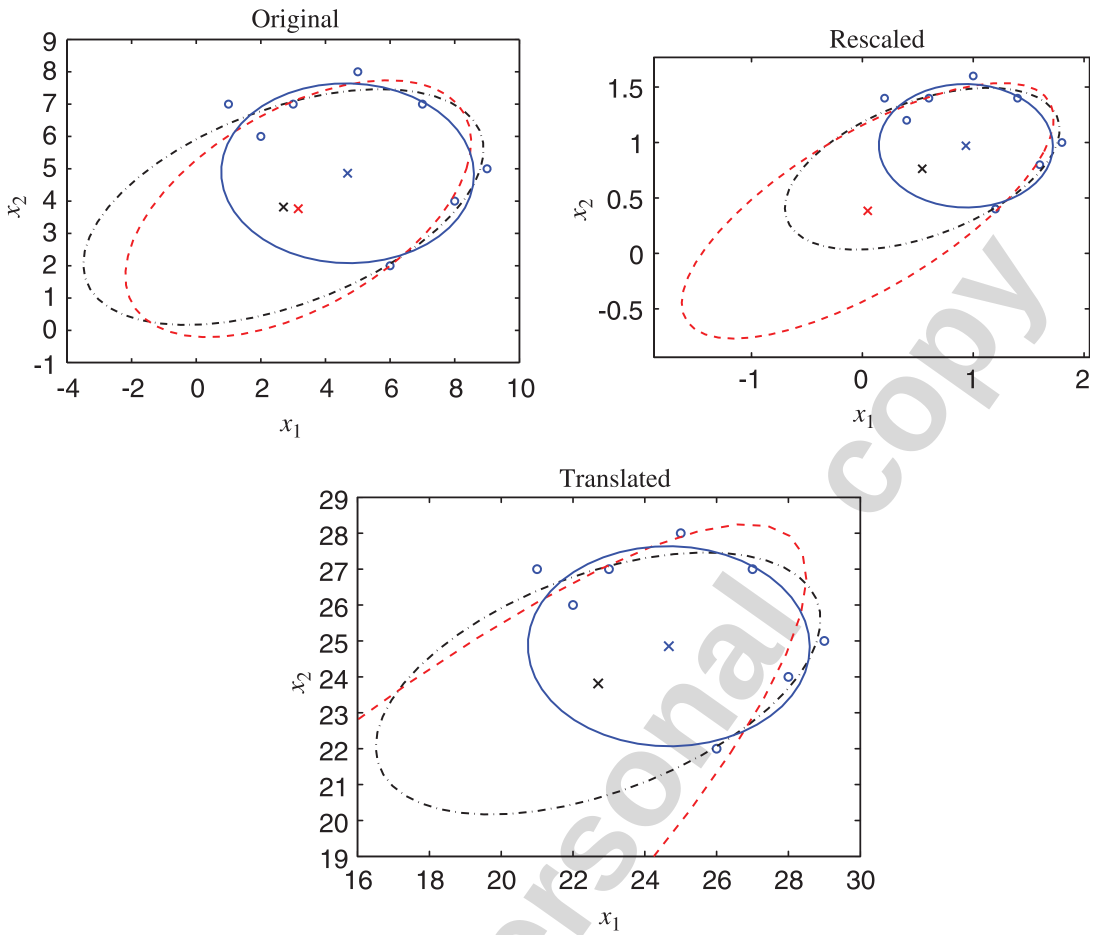

Fig. 1. Simulation example showing the invariance properties of the ALS1, ALS2, and orthogonal regression estimators. Dashed line—ALS1, solid line—ALS2, dashed dotted line—orthogonal regression, ○—data points, ×—centers.

Remark 32. Let $T(x) = kSx + h$ with an orthogonal matrix S, and $k \neq 0$. Then for any sample X,

$$D(T(X)) = k^2 D(X).$$

In the next remark the similarity-invariance of the TALS estimator is concerned.

Remark 33. Consider the transformation T from Remark 32. Denote the set of all the estimated link functions which are solutions to (43) by $\mathrm{Sol}_{\sigma^2}(X)$. Let $\Phi_1 \in \mathrm{Sol}_{\sigma^2}(X)$. Then $\Phi_1 \circ T^{-1}$ is a solution to the problem (51), which is equivalent to

$$\begin{cases} \sum_{l=1}^{m} \Phi^2 * p_{-k^2\sigma^2}(T(x_l)) \to \min, \\ kv_1(\Phi) = 1. \end{cases}$$

Then $\frac{1}{k}\Phi \circ T^{-1} \in \mathrm{Sol}_{k^2\sigma^2}(T(X))$, and $\Phi_1(x)=0 \Leftrightarrow \frac{1}{k}\Phi_1(T^{-1}(T(x)))=0$.

Hence, to introduce the similarity invariance, one has to take the rescaling of measurement error variance into account, and modify Definition 25.

We illustrate the invariance properties of the ALS1, ALS2, and orthogonal regression estimators via a simulation example. The plots on Fig. 1 show data points and the estimates obtained by the

Transformations	ALS1	TALS	ALS2
Isometries preserving the origin	Invariant	Invariant	Invariant
Translations	Not invariant	Invariant	Invariant
Homotheties with the center in the origin	Not invariant	Invariant if σ^2 is rescaled	Invariant

three estimators for the original data (example "special data" from [3]), for the data scaled by factor 0.2 and for the data translated by (20, 20). We see that the ALS2 and orthogonal regression estimators are translation invariant and scale invariant, while the ALS1 estimator is not.

In the next table (see above) it is summarized whether an estimation problem is invariant for any sample X against all transformations within a group.

6. Conclusion

We considered the implicit quadratic measurement error model in a Euclidean space, with normal errors. For the case of known variance, the similarity invariant version of the ALS estimator was presented and its strong consistency was shown. For the case of unknown variance, the consistency of the ALS2 estimators for the surface and the variance were proved under rather mild conditions. The ALS2 estimators are shown to be similarity invariant. We intend to generalize the results for unspecified error distributions.

Acknowledgements

The authors are grateful to two anonymous referees for valuable comments. Research supported by Research Council KUL: GOA-Mefisto 666, IDO /99/003 and /02/009 (Predictive computer models for medical classification problems using patient data and expert knowledge), several PhD/postdoc & fellow grants; Flemish Government: FWO: PhD/postdoc grants, projects, G.0078.01 (structured matrices), G.0407.02 (support vector machines), G.0269.02 (magnetic resonance spectroscopic imaging), G.0270.02 (nonlinear Lp approximation), research communities (ICCoS, ANMMM); IWT: PhD grants; Belgian Federal Science Policy Office IUAP P5/22 ('Dynamical Systems and Control: Computation, Identification and Modelling'); EU: PDT-COIL, BIOPATTERN, ETUMOUR. Part of the research was carried out when Sergiy Shklyar was visiting K.U. Leuven in 2003.

Appendix A. Proofs using the matrix representation of q_{als}

Proposition 34. *The quadratic form* q_{als} *defined by* (9) *is a solution to* (8).

Before the proof of this proposition we consider the following identity.

Lemma 35. *For* $x \sim N(\bar{x}, \sigma^2 I)$, $A \in \mathbb{S}$, $b \in \mathbb{R}^n$

$$\operatorname{var}(x^\top Ax + b^\top x) = 2\sigma^4 \|A\|_F^2 + \sigma^2 \|2A\bar{x} + b\|^2.$$

Proof. There exists a unique decomposition

$$b = 2Ax_1 + b_2, \quad x_1 \in \mathbb{R}^n, \quad b_2 \in \mathbb{R}^n, \quad b_2^\top A = 0.$$

Then

$$x^\top Ax + b^\top x = (x + x_1)^\top A(x + x_1) + b_2^\top x - x_1^\top Ax_1.$$

Since the random vector $\binom{Ax}{b_2^\top x}$ is normally distributed and $\operatorname{cov}(Ax, b_2^\top x) = A\,(\sigma^2 I)\,b_2 = 0$, the random vector Ax and the random variable $b_2^\top x$ are independent. Therefore the random variables $(x + x_1)^\top A(x + x_1) = (Ax)^\top (A^+(Ax) + 2x_1) + x_1^\top Ax_1$ and $b_2^\top x$ are independent. Here A^+ is the pseudoinverse of A. By [8, Theorem 1.8, Corollary 1]

$$\operatorname{var}(x + x_1)^\top A(x + x_1) = 2\sigma^4\|A\|_F^2 + 4\sigma^2\|A(\bar{x} + x_1)\|^2.$$

Finally,

$$\begin{aligned}\operatorname{var}(x^\top Ax + b^\top x) &= \operatorname{var}(x + x_1)^\top A(x + x_1) + \operatorname{var} b_2^\top x \\ &= 2\sigma^4\|A\|_F^2 + \sigma^2\|2A\bar{x} + 2Ax_1\|^2 + \|b_2\|^2 \\ &= 2\sigma^4\|A\|_F^2 + \sigma^2\|2A\bar{x} + b\|^2.\end{aligned}$$

Here we used that $b_2^\top \cdot (2A\bar{x} + 2Ax_1) = 0$. □

Proof of Proposition 34. Let $x \sim N(\bar{x}, \sigma^2 I)$. By [8, Theorem 1.7],

$$\mathbb{E}(x^\top Ax + b^\top x + d - \sigma^2 \operatorname{tr} A) = \bar{x}^\top A\bar{x} + b^\top \bar{x} + d.$$

By Lemma 35,

$$\operatorname{var}(x^\top Ax + b^\top x + d - \sigma^2 \operatorname{tr} A) = \sigma^2\|2A\bar{x} + b\|^2 + 2\sigma^4\|A\|_F^2.$$

Also,

$$\mathbb{E}\|2Ax + b\|^2 = \|2A\bar{x} + b\|^2 + 4\sigma^2\|A\|_F^2.$$

These equalities imply (8) with q_{als} defined by (9). □

Proof of Lemma 7. Consider an entry $Kf_{ijpq}(x)$ of the matrix $[\psi_{\text{als}}(x)]$. Here K is a constant equal to either 1, $\sqrt{2}$, or 2; and f_{ijpq} is a polynomial such that

$$\mathbb{E} f_{ijpq}(\bar{x} + \tilde{x}) = \bar{x}_i\,\bar{x}_j\,\bar{x}_p\,\bar{x}_q, \quad \tilde{x} \sim N(0, \sigma^2 I), \;\; \bar{x} \in \mathbb{R}^n,$$

where $\bar{x}_i$ the ith entry of vector $\bar{x}$. Thus, $f_{ijpq}(x)$ is a homogeneous polynomial in σ and in the entries of x of order 4, and it is an even function in σ. Its variance $\operatorname{var} f_{ijpq}(\bar{x} + \tilde{x})$ is a homogeneous polynomial in σ and in the entries of $\bar{x}$ without σ-free and odd-in-σ monomials. Then $\operatorname{var} f_{ijpq}(\bar{x} + \tilde{x}) \leqslant \text{const}(\sigma^8 + \sigma^2\|\bar{x}\|^6)$. By condition (iv-),

$$\sum_{l=1}^{\infty} \frac{1}{l^2} \operatorname{var} f_{ijpq}(x_l) < \infty.$$

Similar bounds can be found for the variances of other entries of the matrix $\psi_{\text{als}}(x)$. By the strong law of large numbers, $\frac{1}{m}(\Psi_{\text{ols}} - \overline{\Psi}_{\text{ols}}) \to 0$, as $m \to \infty$, a.s. □

Appendix B. Tedious proofs

Proof of Lemma 9. By (22) and (23) from the proof of Lemma 8,

$$\left\| \frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix} - \frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} \bar{x}_l \bar{x}_l^\top & \bar{x}_l \\ \bar{x}_l^\top & 1\end{pmatrix} \right\| \to 0 \quad \text{as } m \to \infty \text{ a.s.}$$

Since for symmetric matrices A and B

$$|\lambda_{\min}(A) - \lambda_{\min}(B)| \leqslant \max_{\|x\|=1} |x^\top (A-B)x| = \|A-B\|,$$

we have

$$\lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\right) - \lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} \bar{x}_l \bar{x}_l^\top & \bar{x}_l \\ \bar{x}_l^\top & 1\end{pmatrix}\right) \to 0,$$

as $m \to \infty$, a.s. Using Lemma 3, one has

$$\liminf_{m\to\infty} \lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\right) \geqslant \varepsilon_0 \|\bar{A}\|^2.$$

Denote $x_{\mathrm{ce}} = \frac{1}{m}\sum_{l=1}^{m} x_l$. Note that x_{ce} depends on m. Then

$$\begin{aligned}\frac{1}{m}\sum_{l=1}^{m} x_l x_l^\top - x_{\mathrm{ce}} x_{\mathrm{ce}}^\top &= \frac{1}{m}\sum_{l=1}^{m}(x_l - x_{\mathrm{ce}})(x_l - x_{\mathrm{ce}})^\top \\ &= \frac{1}{m}\begin{pmatrix} I \\ -x_{\mathrm{ce}}^\top\end{pmatrix}^\top \sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\begin{pmatrix} I \\ -x_{\mathrm{ce}}^\top\end{pmatrix} + \sigma^2 I.\end{aligned}$$

For all $b \in \mathbb{R}^n$,

$$\begin{aligned} & b^\top\left(\frac{1}{m}\sum_{l=1}^{m} x_l x_l^\top - x_{\mathrm{ce}} x_{\mathrm{ce}}^\top\right) b \\ &= \frac{1}{m}\begin{pmatrix} b \\ -x_{\mathrm{ce}}^\top b\end{pmatrix}^\top \sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\begin{pmatrix} b \\ -x_{\mathrm{ce}}^\top b\end{pmatrix} + \sigma^2\|b\|^2 \\ &\geqslant (\|b\|^2 + (x_{\mathrm{ce}}^\top b)^2)\lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\right) + \sigma^2\|b\|^2 \\ &\geqslant \left(\lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\right) + \sigma^2\right)\|b\|^2.\end{aligned}$$

Hence almost surely

$$\liminf_{m\to\infty} \lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m} x_l x_l^\top - x_{\rm ce} x_{\rm ce}^\top\right)$$
$$\geqslant \liminf_{m\to\infty} \lambda_{\min}\left(\frac{1}{m}\sum_{l=1}^{m}\begin{pmatrix} x_l x_l^\top - \sigma^2 I & x_l \\ x_l^\top & 1\end{pmatrix}\right) + \sigma^2 \geqslant \sigma^2 + \varepsilon_0 \|\bar{A}\|^2. \qquad \square$$

Proof of Lemma 10. Let $\bar{c}$ be the center of the true conic, see (24). Denote

$$\bar{y}_l := \bar{x}_l - \bar{c}.$$

Then

$$\begin{aligned} q_{10}((A, b, d), \bar{x}_l) &= 2((\bar{y}_l+\bar{c})^\top A(\bar{y}_l+\bar{c}) + b^\top(\bar{y}_l+\bar{c}) + d)\,\mathrm{tr}\, A + \|2A(\bar{y}_l+\bar{c}) + \bar{b}\|^2 \\ &= 2(\bar{y}_l^\top A\bar{y}_l + (b+2A\bar{c})^\top \bar{y}_l + d + b^\top \bar{c} + \bar{c}^\top A\bar{c})\,\mathrm{tr}\, A \\ &\quad + \|2A\bar{y}_l + \bar{b} + 2A\bar{c}\|^2 \\ &= q_{10}(R(A, b, d), \bar{y}_l), \end{aligned}$$

where R is a linear operator on $\mathbb{V}$ that depends only on $\bar{\beta}$:

$$R(A, b, d) = (A, b + 2A\bar{c}, d + b^\top \bar{c} + \bar{c}^\top A\bar{c}).$$

Then

$$\|R\beta\| \leqslant \|R\|\,\|\beta\| \quad \text{for } \beta \in \mathbb{V}.$$

By Corollary 4,

$$\sum_{l=1}^{m} \|\bar{y}_l\|^2 \geqslant mn\varepsilon_0 \|A\|_F^2 \quad \text{for all } m \geqslant m_0.$$

Then for $\beta = (A, b, d) \in \mathbb{V}$, $R\beta = (A, b_Y, d_Y)$, $y = x - \bar{c}$

$$\begin{aligned} q_{10}(\beta, x) &= 2\langle (yy^\top, y, 1), R\beta\rangle\,\mathrm{tr}\, A + \|2Ay + b_Y\|^2 \\ &\leqslant 2(\|y\|^2+\|y\|+1)\,\|R\beta\|^2\sqrt{n} + (4\|y\|^2+4\|y\|+1)\max(\|A\|^2, \|b_Y\|^2) \\ &\leqslant (3\sqrt{n}+5)(\|y\|^2+1)\|R\beta\|^2, \end{aligned}$$

because

$$|\mathrm{tr} A| \leqslant \sqrt{n}\|A\|_F \leqslant \sqrt{n}\|R\beta\|,$$

$$\|(yy^\top, y, 1)\| \leqslant \|y\|^2 + \|y\| + 1 \leqslant \tfrac{3}{2}(\|y\|^2 + 1),$$

$$4\|y\|^2 + 4\|y\| + 1 \leqslant 5\|y\|^2 + 5,$$

$$\max(\|A\|^2, \|b_Y\|^2) \leqslant \|R\beta\|^2,$$

and of the Schwarz inequality. Then for all $m \geqslant m_0$, $\beta \in \mathbb{V}$, $\|\beta\| = 1$,

$$\begin{aligned}\bar{Q}_{10}(\beta) &\leqslant \sum_{l=1}^{m}(3\sqrt{n}+5)(\|\bar{y}_l\|^2+1)\|R\beta\|^2 \\ &\leqslant (3\sqrt{n}+5)\|R\|^2\left(m+\sum_{l=1}^{m}\|\bar{y}_l\|^2\right) \\ &\leqslant (3\sqrt{n}+5)\|R\|^2\left(\frac{1}{\varepsilon_0\|\bar{A}\|_F^2 n}+1\right)\sum_{l=1}^{m}\|\bar{y}_l\|^2. \end{aligned} \tag{B.1}$$

By (3) and (24),

$$\bar{Q}_{10}(\bar{\beta}) = 4\sum_{l=1}^{m}\|\bar{A}y_l\|^2 \geqslant \frac{4}{\|\bar{A}^{-1}\|^2}\sum_{l=1}^{m}\|\bar{y}_l\|^2. \tag{B.2}$$

By (B.1) and (B.2), inequality (25) holds with

$$C_2 := \frac{3\sqrt{n}+5}{4}\|R\|^2\|\bar{A}^{-1}\|^2\left(\frac{1}{\varepsilon_0\|\bar{A}\|_F^2 n}+1\right). \qquad \square$$

Appendix C. On the generalized eigenvalue problem

C.1. Trigonometry

For $a,\ b \in \mathbb{R}^d$, $a \neq 0$, $b \neq 0$, denote the length the projection of the vector $\frac{a}{\|a\|}$ onto the orthogonal complement to b by $\sin\angle(a,b)$. The following formulae hold:

$$\sin\angle(a,b) = \sqrt{1-\frac{(a^\top b)^2}{\|a\|^2\|b\|^2}},$$

$$\sin\angle(a,b) = \frac{1}{2}\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|\left\|\frac{a}{\|a\|}+\frac{b}{\|b\|}\right\|. \tag{C.1}$$

For $z_1,\ z_2 \in \mathbb{R}^d$, $\|z_1\| = \|z_2\| = 1$, we have

$$\min\{\|z_1-z_2\|, \|z_1+z_2\|\} \leqslant \sqrt{2}\sin\angle(z_1,z_2). \tag{C.2}$$

For vectors $z_1,\ z_2 \in \mathbb{R}^d$, $\|z_1\| = \|z_2\| = 1$,

$$\sin\angle(z_1,z_2) = \frac{\|z_1-z_2\|\sqrt{4-\|z_1-z_2\|^2}}{2} = \frac{\|z_1+z_2\|\sqrt{4-\|z_1+z_2\|^2}}{2}. \tag{C.3}$$

This holds true due to identities (C.1) and $\|z_1-z_2\|^2+\|z_1+z_2\|^2 = 4$.

Let A be a $d \times d$ singular positive semidefinite matrix, let $\lambda_k = \lambda_k(A)$, $k = 1, \dots, d$ be its eigenvalues in ascending order ($\lambda_1 = 0$), and let $b_1, \dots, b_d$ be corresponding eigenvectors forming an orthogonal normalized basis of $\mathbb{R}^d$. Suppose $x = \sum_{i=1}^{d} \alpha_i b_i \neq 0$. Then $\|x\|^2 \sin^2 \angle(x, b_1) = \sum_{i=2}^{d} \alpha_i^2$. Hence for all $x \in \mathbb{R}^n$, $x \neq 0$,

$$x^\top A x \geqslant \|x\|^2 \lambda_2 \sin^2 \angle(x, b_1). \tag{C.4}$$

For all $a, b \in \mathbb{R}^n$, $a \neq 0$, the inequality

$$\left(\|a\| - \frac{(a,b)}{\|a\|} \right)^2 \geqslant 0$$

implies

$$\|b\|^2 - \frac{(a,b)^2}{\|a\|^2} \leqslant \|a\|^2 - 2(a,b) + \|b\|^2.$$

Extracting the root, we get for $b \neq 0$ that

$$\|b\| \sin \angle(a, b) \leqslant \|a - b\|. \tag{C.5}$$

Lemma 36. *For any* $a, b, c \in \mathbb{R}^d \setminus \{0\}$,

$$\sin \angle(a, b) \leqslant \sin \angle(a, c) + \sin \angle(c, b).$$

Proof.

$$r(a, b) := \min \left\{ \left\| \frac{a}{\|a\|} - \frac{b}{\|b\|} \right\|, \left\| \frac{a}{\|b\|} + \frac{b}{\|b\|} \right\| \right\}$$

is a pseudometric on $\mathbb{R}^d \setminus \{0\}$. For all $a \neq 0$ and $b \neq 0$, $0 \leqslant r(a, b) \leqslant \sqrt{2}$.

Consider the function $f(t) = \frac{1}{2} t \sqrt{4 - t^2}$. Since $f(0) = 0$ and $f(t)$ is increasing and concave on the interval $[0, \sqrt{2}]$, therefore $f(r(a, b))$ is a pseudometric as well. By (C.3), $\sin \angle(a, b) = f(r(a, b))$. The inequality holds true. $\square$

Remark 37. We define sine between elements of the Euclidean space $\mathbb{V}$ by

$$\sin \angle(\beta_1, \beta_2) := \sqrt{1 - \frac{\langle \beta_1, \beta_2 \rangle^2}{\|\beta_1\|^2 \|\beta_2\|^2}}, \quad \beta_1, \beta_2 \in \mathbb{V} \setminus \{0\}.$$

The properties proved in this subsection for vectors in $\mathbb{R}^d$ remain true. Inequality (C.4) changes to the following. Let $Q(\beta)$ be a positive semidefinite quadratic form on $\mathbb{V}$, and Ψ be a corresponding self-adjoint operator. Let $\beta_0 \neq 0$ and $Q(\beta_0) = 0$. Then

$$Q(\beta) \geqslant \|\beta\|^2 \lambda_2(\Psi) \sin^2 \angle(\beta, \beta_0), \quad \beta \in \mathbb{V}, \ \beta \neq 0. \tag{C.6}$$

C.2. Definite matrix pairs

Let A and B be $d \times d$ matrices. A number λ is called a (generalized) eigenvalue of A w.r.t. B if the matrix $A - \lambda B$ is singular. Infinity ∞ is called a generalized eigenvalue if the matrix B is singular.

The set $\{x : Ax = \lambda Bx\}$ (or $\{x : Bx = 0\}$ if $\lambda=\infty$) is called the eigenspace corresponding to the eigenvalue λ. Its dimension is called the geometric multiplicity of the eigenvalue λ, and its nonzero elements are called generalized eigenvectors.

A pair of real symmetric $d \times d$ matrices is called definite if

$$\gamma(A, B) := \min_{\|x\|=1} \sqrt{(x^\top Ax)^2 + (x^\top Bx)^2} > 0.$$

As $\gamma(A, B) = \min_{\|x\|=1} |x^\top (A + iB)x|$, therefore

$$|\gamma(A_1, B) - \gamma(A_2, B)| \leqslant \max_{\|x\|=1} |x^\top (A_1 - A_2)x| = \|A_1 - A_2\|. \tag{C.7}$$

If a matrix pair is definite, then

- There exist real α and β, such that the matrix $\alpha A + \beta B$ is positive definite, whence $-\beta/\alpha$ is not a generalized eigenvalue, see [10, Theorem IV.1.18].
- All the finite generalized eigenvalues are real.
- The sum of the geometric multiplicities of all the generalized eigenvalues is equal to d.

We enumerate the generalized eigenvalues according to the following order. Let $\alpha A + \beta B$ be positive definite. At first, we count the generalized eigenvalues from interval from $-\beta/\alpha$ down to $-\infty$ in decreasing order, repeating each eigenvalue according to its multiplicity. Then we count ∞ (repeating it $\dim \operatorname{Ker} B$ times). At last, we count the generalized eigenvalues from $+\infty$ down to $-\beta/\alpha$.

(If $\alpha = 0$, then B is definite and all generalized eigenvalues are finite. We enumerate them in decreasing order.)

This enumeration does not depend on the choice of α and β satisfying $\alpha A + \beta B > 0$. The enumeration coincides with the ordering given in [10, pp. 313–314].

If $\alpha A+\beta B$ is a semidefinite matrix, then the properties stated above hold true with the following exception. $-\beta/\alpha$ can be a generalized eigenvalue of A w.r.t. B, and then it lies either at the beginning or at the end of the enumeration.

A chordal distance is a metric on $\mathbb{R} \cup \{\infty\}$, defined by relations

$$\chi(\lambda, \mu) := \frac{|\lambda - \mu|}{\sqrt{1+\lambda^2}\sqrt{1+\mu^2}}, \quad \chi(\lambda, \infty) := \frac{1}{\sqrt{1+\lambda^2}}.$$

A chordal distance between definite matrix pairs (A, B) and (C, D) is

$$\rho_D[(A, B), (C, D)] := \max_{\|x\|=1} \frac{|x^\top Ax\, x^\top Dx - x^\top Cx\, x^\top Bx|}{\sqrt{(x^\top Ax)^2 + (x^\top Bx)^2}\sqrt{(x^\top Cx)^2 + (x^\top Dx)^2}}.$$

References

[1] C.-L. Cheng, H. Schneeweiss, Polynomial regression with measurement errors, J. Roy. Statist. Soc. Ser. B 60 (1998) 189–199.

[2] W. Fuller, Measurement Error Model, Wiley, New York, 1987.

[3] W. Gander, G. Golub, R. Strebel, Fitting of circles and ellipses: least squares solution, BIT 34 (1994) 558–578.

[4] K. Kanatani, Statistical bias of conic fitting and renormalization, IEEE Trans. Pattern Anal. Machine Intell. 16 (3) (1994) 320–326.
[5] A. Kukush, I. Markovsky, S. Van Huffel, Consistent estimation in an implicit quadratic measurement error model, Comput. Statist. Data Anal. 47 (1) (2004) 123–147.
[6] L. Lehmann, Theory of Point Estimation, Wiley, New York, 1983.
[7] I. Markovsky, A. Kukush, S. Van Huffel, Consistent least squares fitting of ellipsoids, Numer. Math. 98 (1) (2004) 177–194.
[8] G.A.F. Seber, Linear Regression Analysis, Wiley, New York, 1977.
[9] A.N. Shiryaev, Probability, Springer, New York, 1996.
[10] G. Stewart, J. Sun, Matrix Perturbation Theory, Academic Press, London, 1990.