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Efficiency of maximum likelihood estimators in Nevzorov's record model

A statistical model to analyse claims arising out of natural catastrophes is presented. Basing on record values, an exponential trend over time is detected and efficiency of the maximum likelihood estimator of trend parameter is shown. The 3-parameter model involving such a trend is proposed. It interprets the observed claims as a stochastically increasing sequence of Fréchet distributed random variables. Efficiency of the joint maximum likelihood estimator is shown.

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1 Introduction

It is evident that insurance claims due to the occurrence of natural catastrophes have raised enormously over the past decades all over the world. In Pfeifer (1997), and later in Kukush (1999) and Kukush, Chernikov (2001) a particular approach to the investigation of catastrophe claims in the presence of a trend was presented, which is based on a combination of parametric and semi-parametric methods. In Kukush (1999) some results concerning asymptotic properties of the maximum likelihood estimator (MLE) were stated, and in Kukush and Chernikov (2001) a goodness-of-fit test was obtained.

In this article the efficiency results are proved for the semi-parametric and the three-parametric models. These results were announced in Kukush (1999). We mention that the efficiency of the MLE does not follow from general theory given in Ibragimov and Has'minskii, because we deal with non-i.i.d. observations.

In Section 2 the Nevzorov's record model is introduced. In Section 3 the efficiency result of the semi-parametric MLE is formulated. Section 4 contains the three-parametric model and the corresponding efficiency results. The proofs are given is Section 5, and Section 6 concludes.

2 Nevzorov's record model

A record model has been studied by V.B.Nevzorov(1988) and K.Borovkov and D.Pfeifer(1995). Assume that the yearly catastrophe claims considered here are realizations of an independent sequence

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$\{X_n, n \geq 1\}$ of random variables (r.v) with support $\mathbf{R}^+ := [0, \infty)$ and continuous cumulative d.f $\{F_n, n \geq 1\}$, s.t.

$$F_n = F^{\gamma_n} \text{ , with } \gamma_n := \gamma^{n-1} \text{ , } \gamma \geq 1. \tag{1}$$

Here F is a fixed cumulative d.f. with F(0)=0. Define record indicators by

$$I_1 := 1,\ I_n := \begin{cases} 1, \text{ if } X_n > \max\{X_1, ..., X_{n-1}\} \\ 0, \text{ otherwise} \end{cases} \text{ for } n \geq 2,$$

i.e. $I_n = 1$ iff observation X_n is a record value in the sequence. Under the above assumptions, the record indicators are independent r.v. with

$$p_n(\gamma) := P_\gamma(I_n = 1) = \frac{\gamma_n}{\gamma_1 + ... + \gamma_n} = \frac{1}{1 + \gamma^{-1} + ... + \gamma^{-n+1}}.$$

Consider also the number S_n of record values in a finite number of observations,

$$S_n := \sum_{i=1}^{n} I_i,\ n \geq 1.$$

The record times $T_1, ...T_{S_n}$ denote the observation times at which record values occur:

$$T_1 := 1,\ T_{k+1} := min\{i \leq n | X_i > X_{T_k}\},\ 1 \leq k < S_n.$$

The unknown parameter γ, see (1), is called trend parameter. If $\gamma = 1$, then we have the i.i.d situation **(no trend)** , while for $\gamma > 1$, the r.v. $\{X_n\}$ are stochastically increasing **(positive trend)** . Given the observations $I_1, ...I_n,\ n \geq 2$,of record indicators in a sequence of data, the log-likelihood function $L(\gamma)$ for $\gamma \geq 1$ is given by

$$L(\gamma) = \ln\left(\prod_{i=2}^{n} p_i(\gamma)^{I_i}\left(1 - p_i(\gamma)\right)^{1-I_i}\right) = \sum_{i=2}^{n} I_i \ln(p_i(\gamma)) + \sum_{i=2}^{n}(1 - I_i)\ln(1 - p_i(\gamma)). \tag{2}$$

For $\gamma > 1$ it is possible to rewrite it in a way, which is more comfortable for numerical optimization:

$$L(\gamma) = S_n \ln(\gamma - 1) - \ln(\gamma^n - 1) - \sum_{k=2}^{S_n} \ln(1 - \gamma^{1-T_k}). \tag{3}$$

The semi-parametric MLE $\hat{\gamma} = \hat{\gamma}_n$ is defined as a measurable function of $I_1, ..., I_n$, for which

$$\hat{\gamma} \in \arg\max_{\gamma \geq 1} L(\gamma). \tag{4}$$

If $I_1, ..., I_n \neq (1, 1, ..., 1)$, then maximum in (4) is attained. Otherwise maximum in (4) is not attained, and in that case we set $\hat{\gamma} := +\infty$. It happens with probability tending to zero as $n \to +\infty$.

In Kukush (1999) the following Theorem was formulated for the semi-parametric MLE:

Theorem 1. *Let $\gamma > 1$. Then the MLE $\hat{\gamma}$ is asymptotically normal, namely the normalized estimator $\sqrt{n}(\hat{\gamma}_n - \gamma)$ converges in distribution to a normal law with mean 0 and variance $\sigma_\infty^2 = \gamma^2(\gamma - 1)$.*

3 Efficiency of semi-parametric MLE

We shall understand efficiency in the sense of Hajék bound, see I. Ibragimov and R. Has'minskii (1981). Introduce the class $W_{e,2}$ of bell-shaped loss functions. These functions w(u), $u \in \mathbf{R}$, satisfy the following conditions:

a) $\mathrm{w}(u) \geq 0, u \in \mathbf{R}$; $\mathrm{w}(0) = 0$, w is continuous at u=0 and is not identically 0.

b) w is even function.

c) w is nondecreasing for $u \geq 0$.

d) The growth of w as $u \to +\infty$ is slower than any one of the functions $\exp(\varepsilon u^2)$, $\varepsilon > 0$.

Denote by ξ a standard Gaussian r.v.

Theorem 2. *Let $\gamma_0 > 1$, and the function $w : \mathbf{R} \to \mathbf{R}$ be bounded, Borel measurable and continuous a.e. with respect to Lebesgue measure. Then*

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \sup_{\gamma : |\gamma - \gamma_0| < \delta} E_\gamma w\left(\sqrt{\frac{n}{\gamma_0 - 1}} \cdot \frac{\hat{\gamma}_n - \gamma}{\gamma_0} \right) = Ew(\xi). \tag{5}$$

Theorem 3. *Let $\gamma_0 > 1$. Then for any family γ_n^* of estimators of γ, based on the observations $I_1, ..., I_n$, and for any loss function $w \in W_{e,2}$, the inequality holds:*

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \sup_{\gamma : |\gamma - \gamma_0| < \delta} E_\gamma w\left(\sqrt{\frac{n}{\gamma_0 - 1}} \cdot \frac{\hat{\gamma}_n - \gamma}{\gamma_0} \geq Ew(\xi) \right). \tag{6}$$

The inequality gives a lower bound for the loss of arbitrary normalized estimator. Theorem 2 shows that equality in (6) is attained by the semiparametric MLE. Thus the MLE has asymptotically the smallest possible averaged loss.

4 The three parametric statistical model

Since by economic arguments it is reasonable to assume that a possible trend in the data is of exponential type, we shall base the parametric model on a combination of Nevzorov's record model and the parametric class of Fréchet distributions (one of the extreme-value distribution classes). Thus we assume now that the cumulative d.f. F_n for the yearly claims are of the form

$$F_n(x) = \exp(-\gamma^{n-1}(Ax)^{-\alpha}),\ n = 1, 2, ...,\ x > 0.$$

Here $A > 0, \alpha > 0$ and $\gamma \geq 1$ are parameters of interest.

For the above parametric family, the log-likelihood function $L(A, \alpha, \gamma)$ for the observed data set $X_1, ..., X_n$ is given by

$$L(A, \alpha, \gamma) = \frac{n(n-1)}{2} \ln \gamma - (\alpha + 1) \sum_{i=1}^{n} \ln X_i - \sum_{i=1}^{n} \gamma^{i-1} (AX_i)^{-\alpha} + n \ln(\alpha A^{-\alpha}). \tag{7}$$

Choose a parameter set

$$\Theta = (0, +\infty) \times (0, +\infty) \times [1, +\infty)$$

and define the joint MLE of the parameters of interest as a measurable vector function $(\hat{A}, \hat{\alpha}, \hat{\gamma})$ of $X_1, ..., X_n$, for which $(\hat{A}, \hat{\alpha}, \hat{\gamma}) \in \arg\max_{(A,\alpha,\gamma)\in\Theta} L(A, \alpha, \gamma)$. One can show that maximum here is attained with probability tending to 1 as $n \to \infty$. In Kukush (1999) the following theorem was formulated:

Theorem 4. *If $\gamma > 1$, then the joint MLE is asymptotically normal, namely the normalized estimator*

$$\sqrt{n}(R_n T R_n')^{1/2} \begin{pmatrix} \hat{A} - A \\ \hat{\alpha} - \alpha \\ n(\ln\hat{\gamma} - \ln\gamma) \end{pmatrix}$$

converges in distribution to a normal law with mean 0 and a unit covariance matrix, where

$$R_n := \begin{pmatrix} \frac{\alpha}{A} & 0 & 0 \\ 0 & -\frac{1}{\alpha} & n \cdot \frac{\ln\gamma}{\alpha} \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1-\gamma_e & \frac{1}{2} \\ 1-\gamma_e & \frac{1}{6}\pi^2 + \gamma_e^2 - 2\gamma_e + 1 & \frac{1}{2}(1-\gamma_e) \\ \frac{1}{2} & \frac{1}{2}(1-\gamma_e) & \frac{1}{3} \end{pmatrix} \tag{8}$$

R_n' is R_n transposed, and γ_e is Euler's constant, $\gamma_e \simeq 0.5772$.

To formulate the efficiency result, introduce the class $V_{e,2}$ of loss functions of three variables. These functions $w(u),\ u \in \mathbf{R}^3$, satisfy the following conditions.

a) $w(u) \geq 0,\ u \in \mathbf{R}^3;\ w(0) = 0$; w is continuous at u=0 and is not identically 0; w is Borel measurable.

b) $w(u) = w(-u),\ u \in \mathbf{R}^3$.

c) For each $c > 0$, the set $\{u : w(u) < c\}$ is convex.

d) The growth of w as $\|u\| \to +\infty$ is slower than any one of the functions $\exp(\varepsilon\|u\|^2), \varepsilon > 0$.

Denote by ζ a standard Gaussian random vector in $\mathbf{R}^3$.

Theorem 5. *Let the function $w : \mathbf{R}^3 \to \mathbf{R}$ be bounded, Borel measurable and continuous a.e. with respect to Lebesgue measure, and*

$$A_0 > 0,\ \alpha_0 > 0,\ \gamma_0 > 1. \tag{9}$$

Then

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \sup_{\substack{\beta : |A - A_0| < \delta, \\ |\alpha - \alpha_0| < \delta, \\ |\ln\gamma - \ln\gamma_0| < \delta/n}} E_\beta \left\{ w \left[\sqrt{n} \cdot (R_{n0} T R_{n0}')^{1/2} \begin{pmatrix} \hat{A} - A \\ \hat{\alpha} - \alpha \\ n(\ln\hat{\gamma} - \ln\gamma) \end{pmatrix} \right] \right\} = Ew(\zeta),$$

where T and R_{n0} are given in (8), with $(A, \alpha, \gamma) = (A_0, \alpha_0, \gamma_0)$, and R_{n0}' is R_{n0} transposed.

Theorem 6. *Assume (9). Then for any family $T_n^* = (T_n^1, T_n^2, T_n^3)$ of estimators of $\beta = (A, \alpha, \gamma)'$, based on observations $X_1, ..., X_n$, and for any loss function $w \in V_{e,2}$, the inequality holds:*

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \sup_{\substack{\beta : |A - A_0| < \delta, \\ |\alpha - \alpha_0| < \delta, \\ |\ln\gamma - \ln\gamma_0| < \delta/n}} E_\beta \left\{ w \left[\sqrt{n} \cdot (R_{n0} T R_{n0}')^{1/2} \begin{pmatrix} T_n^1 - A \\ T_n^2 - \alpha \\ n(\ln T_n^3 - \ln\gamma) \end{pmatrix} \right] \right\} \geq Ew(\zeta).$$

We see, that the joint MLE has asymptotically the smallest possible averaged loss.

5 Proofs

Proof of theorem 2. One can show that convergence in the theorem 1 is uniform for $\gamma \in [1+\epsilon, C],\ \epsilon > 0,\ C > 1+\epsilon$. The equality (5) follows now from Ibragimov and Has'minskii (1981), p.177.

Proof of theorem 3.

Introduce the normalized log-likelihood function Q, see (3),

$$Q = Q_n(\gamma) := \frac{1}{n} L(\gamma) = Q_{(1)} + Q_{(2)}, \tag{10}$$

with

$$Q_{(1)} := \frac{1}{n} \sum_{i=2}^{n} (I_i - p_i^0) \ln \frac{p_i}{1-p_i} \tag{11}$$

and

$$Q_{(2)} := \frac{1}{n} \sum_{i=2}^{n} \Big[p_i^0 \ln \frac{p_i}{1-p_i} + \ln (1-p_i) \Big]. \tag{12}$$

(i) *Convergence of the first derivative.* From (11) we have

$$\sqrt{n} Q'_{(1)}(\gamma_0) = \frac{1}{\sqrt{n}} \sum_{i=2}^{n} (I_i - p_i^0) \frac{p'_i(\gamma_0)}{p_i^0(1-p_i^0)}.$$

But $\lim_{i \to \infty} p'_i(\gamma_0) = {\gamma_0}^{-2}$, therefore

$$\mathrm{Var}(\sqrt{n} Q'_{(1)}(\gamma_0)) = \frac{1}{n} \sum_{i=2}^{n} \frac{(p'_i(\gamma_0))^2}{p_i^0(1-p_i^0)} \to \frac{{\gamma_0}^{-4}}{{\gamma_0}^{-1}(1-{\gamma_0}^{-1})} = \frac{1}{{\gamma_0}^2(\gamma_0 - 1)},\ n \to \infty.$$

By CLT in Lyapunov form we get

$$\sqrt{n} Q'_{(1)}(\gamma_0) \to N(0, \frac{1}{{\gamma_0}^2(\gamma_0-1)}) \text{ in distribution.} \tag{13}$$

Now, the function

$$\phi(p_i) := p_i^0 \ln \frac{p_i}{(1-p_i)} + \ln(1-p_i)$$

has minimum point $p_i = p_i^0$, therefore

$$\frac{d}{d\gamma} \phi(p_i) \Big|_{\gamma=\gamma_0} = \frac{d\phi(p_i^0)}{dp_i} \frac{dp_i(\gamma_0)}{d\gamma} = 0$$

and

$$Q'_{(2)}(\gamma_0) = 0. \tag{14}$$

Now, (13) and (14) imply

$$\sqrt{n} Q'_n(\gamma_0) \to N(0, \frac{1}{{\gamma_0}^2(\gamma_0-1)}) \text{ in distribution.} \tag{15}$$

(ii) *Convergence of the second derivative.* We have

$$Q''_{(1)}(\gamma_0) = \frac{1}{n}\sum_{i=2}^{n}(I_i - p_i^0)\Big(\frac{d}{d\gamma}\frac{p_i'}{p_i(1-p_i)}\Big|_{\gamma=\gamma_0}\Big) \tag{16}$$

The derivatives in (16) form a bounded sequence, and using the second moment one can easily show that

$$Q''_{(1)}(\gamma_0) \to 0 \text{ in probability } P_{\gamma_0}. \tag{17}$$

Now,

$$\frac{d^2\phi(p_i)}{d\gamma^2}\Big|_{\gamma=\gamma_0} = \phi''(p_i^0)(p_i'(\gamma_0))^2 + \phi'(p_i^0)p_i''(\gamma_0) = \phi''(p_i^0)(p_i'(\gamma_0))^2,$$

and $\phi''(p_i^0) = -\frac{1}{p_i^0(1-p_i^0)}$. Then

$$\lim_{i\to\infty}\frac{d^2\phi(p_i)}{d\gamma^2}\Big|_{\gamma=\gamma_0} = -\lim_{i\to\infty}\frac{(p_i'(\gamma_0))^2}{p_i^0(1-p_i^0)} = -\frac{1}{{\gamma_0}^2(\gamma_0-1)}.$$

Therefore

$$\lim_{n\to\infty} Q''_{(2)}(\gamma_0) = -\frac{1}{{\gamma_0}^2(\gamma_0-1)}. \tag{18}$$

Finally, (17) and (18) imply the convergence

$$Q''_n(\gamma_0) \to -\frac{1}{{\gamma_0}^2(\gamma_0-1)} \text{ in probability } P_{\gamma_0}. \tag{19}$$

(iii) *Oscillations of Q''_n.* Fix $\epsilon > 0,\ C > 1+\epsilon$. From (2) we get for $\gamma \in [1+\epsilon, C]$

$$Q'''_n(\gamma) = \frac{1}{n}\sum_{i=2}^{n} I_i(\ln p_i)''' + \frac{1}{n}\sum_{i=2}^{n}(\ln(1-p_i))'''(1-I_i),$$

and there exists a constant M, s.t. for all $n \geq 1$, and all $\gamma \in [1+\epsilon, C]$

$$|Q'''_n(\gamma)| \leq M \tag{20}$$

(iv) *Local asymptotic normality (LAN).* Let $\gamma_0 > 1$ be the true value of trend parameter γ. Set $\psi := \gamma_0\sqrt{\gamma_0 - 1}$, let $u \in \mathbf{R}$. Use the Taylor expansion:

$$L(\gamma_0 + \frac{\psi}{\sqrt{n}}u) - L(\gamma_0) = n\Big(Q_n(\gamma_0 + \frac{\psi}{\sqrt{n}}u) - Q_n(\gamma_0)\Big) =$$

$$= \sqrt{n}Q'_n(\gamma_0)\psi u + \frac{1}{2}Q''_n(\gamma_0)\psi^2u^2 + \frac{1}{6}\frac{1}{\sqrt{n}}Q'''_n(\bar{\gamma})\psi^3u^3. \tag{21}$$

Here $\bar{\gamma}$ is an intermediate point between γ_0 and $\gamma_0 + \frac{\psi}{\sqrt{n}}u$. Now, by (15), (19) and (20), we have

$$\Delta_n := \sqrt{n}Q'_n(\gamma_0)\psi \to N(0;1) \text{in distribution}, \tag{22}$$

$$\frac{1}{2}Q''_n(\gamma_0)\psi^2 = \frac{1}{2} + o_p(1),\ n \to \infty, \tag{23}$$

$$\frac{1}{6}\frac{1}{\sqrt{n}}Q'''_n(\bar{\gamma})\psi^3u^3 \to 0,\ n \to \infty \text{ a.s.} \tag{24}$$

From (21)-(24) we have the LAN property

$$L(\gamma_0 + \frac{\psi}{\sqrt{n}} u) - L(\gamma_0) = \Delta_n u - \frac{1}{2} u^2 + o_p(1),$$

with $\Delta_n \to N(0,1)$ in distribution. Therefore, (see Ibragimov, Has'minskii (1981)) for $w \in W_{e,2}$ and arbitrary family γ^* of estimators

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \sup_{\gamma : |\gamma - \gamma_0| < \delta} E_\gamma w\Big(\frac{\sqrt{n}}{\psi}(\gamma^* - \gamma)\Big) \geq Ew(\xi),$$

where ξ is a standard Gaussian r.v.

Proof of Theorem 5.

For fixed $0 < A_1 < A_2,\ 0 < \alpha_1 < \alpha_2,\ R > 0$ define the set

$$M := (A_1, A_2) \times (\alpha_1, \alpha_2) \times (0, R). \tag{25}$$

From the Theorem 4 it follows that

$$\sqrt{n}(R_0 T R_0')^{1/2} \begin{pmatrix} \hat{A} - A_0 \\ \hat{\alpha} - \alpha_0 \\ n(\ln \hat{\gamma} - \ln \gamma_0) \end{pmatrix} \to \zeta \sim N(0, I_3) \tag{26}$$

in law as $n \to \infty$. Here $\psi_0 = n \ln \gamma_0,\ \hat{\psi} = n \ln \hat{\gamma}$,

$$R_0 = \begin{pmatrix} \frac{\alpha_0}{A_0} & 0 & 0 \\ 0 & -\frac{1}{\alpha_0} & \frac{\psi_0}{\alpha_0} \\ 0 & 0 & -1 \end{pmatrix}.$$

Convergence in (26) holds uniformly in law for $l_0 \in M$ (see the definition of uniform convergence in law in Ibragimov and Has'minskii (1981), p.365). Then for the function w indicated before Theorem 5 we have

$$E_{l_o}\Big\{ w\Big[\sqrt{n}(R_0 T R_0')^{1/2} \begin{pmatrix} \hat{A} - A_0 \\ \hat{\alpha} - \alpha_0 \\ \hat{\psi} - \psi_0 \end{pmatrix}\Big]\Big\} \to Ew(\zeta),\ n \to \infty,$$

uniformly for $l_0 \in M$. Therefore for each δ, such that $\{l : ||l - l_0|| < \delta\} \subset M$,

$$\lim_{n \to \infty} \sup_{||l - l_0|| < \delta} E_l\Big\{ w\Big[\sqrt{n}(R(l) T R'(l))^{1/2}(\hat{l} - l)\Big]\Big\} = Ew(\zeta).$$

Here

$$R(l) = \begin{pmatrix} \frac{\alpha}{A} & 0 & 0 \\ 0 & -\frac{1}{\alpha} & \frac{\psi}{\alpha} \\ 0 & 0 & -1 \end{pmatrix}, \quad l = (A, \alpha, \psi)^T. \tag{27}$$

Following the line of Ibragimov and Has'minskii (1981), p.177 we get that

$$\lim_{\delta \to 0} \liminf_{n \to \infty} \sup_{||l - l_0|| < \delta} E_l\Big\{ w\Big[\sqrt{n}(R(l) T R'(l))^{1/2}(\hat{l} - l)\Big]\Big\} = Ew(\zeta),$$

which implies Theorem 5.

Proof of Theorem 6. We apply the results of Strasser (1996) on local asymptotic normality for non-i.i.d. observations. We write the Fréchet d.f. in a form

$$F_l(t) = \exp(-e^{\psi}(At)^{-\alpha}),\ t > 0,$$

where $l = (A, \alpha, \psi),\ A, \alpha > 0,\ \psi \geq 0$. Denote by P_l the measure with d.f. F_l, P_l is defined on the σ-field $S(\mathbf{R})$ of Lebesgue measurable sets. Let μ be a restriction of Lebesgue measure on $\mathbf{R}^+$, $\mu(A) = \mathrm{mes}(A \bigcap \mathbf{R}^+),\ A \in S(\mathbf{R})$. For the density $\rho_l(t) = \frac{dP_l}{d\mu}(t)$ we have

$$\ln \rho_l(t) = \psi + \ln \alpha - \alpha \ln A - (\alpha + 1) \ln t - (At)^{-\alpha} e^{\psi},\ t > 0.$$

Now, the column vector of derivatives equals

$$L_l(t) = \Big(\frac{\partial}{\partial A}, \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \phi}\Big)' \ln \rho_l(t) = \begin{pmatrix} -\frac{\alpha}{A} + \alpha A^{-\alpha-1} t^{-\alpha} e^{\psi} \\ \frac{1}{\alpha} - \ln A - \ln t + e^{\psi}(At)^{-\alpha} \ln(At) \\ 1 - (At)^{-\alpha} e^{\psi} \end{pmatrix}.$$

Let Z_l be a r.v. distributed with the law $F_l(t)$, and Z has standard Fréchet distribution with $l = (1, 1, 0)$. Then $Z_l \stackrel{d}{=} A^{-1} e^{\frac{\psi}{\alpha}} Z^{\frac{1}{\alpha}},\ Z \stackrel{d}{=} (AZ_l)^{\alpha} e^{-\psi}$, and

$$L_l(Z_l) \stackrel{d}{=} \begin{pmatrix} \frac{\alpha}{A}(Z^{-1} - 1) \\ \frac{1}{\alpha}(-\ln Z + Z^{-1} \ln Z + \psi Z^{-1} - E(-\ln Z + Z^{-1} \ln Z + \psi Z^{-1})) \\ 1 - Z^{-1} \end{pmatrix}.$$

According to the definition 4.1 of Strasser (1996), p.892, the family of measures $\{P_l :\ l \in M\}$is uniformly continuously L^2-differentiable on each set M of the form (25). Let $\{l_{ni}, t_{ni}\} \subset M$,

$$l_{ni} = \left(A, \alpha, \frac{i-1}{n}\psi\right),\ t_{ni} = \left(t_1, t_2, \frac{i-1}{n} t_3\right),\ 1 \leq i \leq n.$$

The conditions of Theorem 4.9 from Strasser (1996) hold. Now, for $t = (t_1, t_2, t_3)$

$$t_{ni} \cdot L_{l_{ni}}(Z_i) = t \cdot R(l) \cdot \Lambda_i \zeta_i,$$

where $R(l)$ is given in (26),

$$\zeta_i := \begin{pmatrix} z_i^{-1} - E z_i^{-1} \\ (1 - z_i^{-1}) \ln z_i - E(1 - z_i^{-1}) \ln z_i \\ \frac{i-1}{n}(z_i^{-1} - E z_i^{-1}) \end{pmatrix} \tag{28}$$

$$\Lambda_i := \mathrm{diag}(1, 1, \frac{i-1}{n}). \tag{29}$$

One can show that

$$\sigma_n^2 := \mathrm{cov}\Big(tR(l)\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \Lambda_i \zeta_i\Big) \to tR(l)TR'(l)t',\ n \to \infty, \tag{30}$$

and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \Lambda_i \zeta_i \to N(0, T) \tag{31}$$

in law. By Theorem 4.9 of Strasser (1996) we have

$$f_n := \frac{d\bigotimes_{i=1}^{n} P_{l_{ni}+t_{ni}/\sqrt{n}}}{d\bigotimes_{i=1}^{n} P_{l_{ni}}} = exp\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} t_{ni}L_{l_{ni}} - \frac{1}{2}\sigma_n^2 + r_n\Big),$$

with $r_n \to 0$ in $\bigotimes_{i=1}^{n} P_{l_{ni}}$. By (30), (31) we get with $R = R(l)$:

$$f_n = \exp\Big(tR\gamma_n - \frac{1}{2}t(RTR')t' + \bar{r}_n\Big), \tag{32}$$

where $\bar{r}_n \to 0$ in $\bigotimes_{i=1}^{n} P_{l_{ni}}$. Denote

$$Q^{(n)}_{(A\alpha\psi)} := \bigotimes_{i=1}^{n} P_{(A,\alpha,\frac{i-1}{n}\psi)}.$$

The observations X_i in Theorem 6 have the distributions $P_{(A,\alpha,\frac{i-1}{n}\psi)}$. Then $Q^{(n)}_{(A\alpha\psi)}$ is a joint distribution of the observations. We set $t = u(RTR')^{1/2}$ and rewrite (32) in a form

$$\frac{dQ^{(n)}_{(A\alpha\psi)+\frac{u}{\sqrt{n}}(RTR')^{-1/2}}}{dQ^{(n)}_{(A\alpha\psi)}}\Big(X_1,\dots,X_n\Big) = \exp(u\cdot\triangle_n - \frac{1}{2}||u||^2 + \bar{r}_n), \tag{33}$$

with $L\Big(\triangle_n|P^{(n)}_{(A\alpha\psi)}\Big) \to N(0, I_3)$. Relation (33) is LAN-property. By Theorem 12.1 of Ibragimov and Has'minskii (1981), p.162, for any family T_n of estimators of parameters A, α, ψ and for any loss function $w \in W_{l,2}$,

$$\lim_{\delta\to 0}\liminf_{n\to\infty}\sup_{||l-l_0||<\delta} E_l\Big\{w\Big[\sqrt{n}(R_0TR_0')^{1/2}(T_n - l)\Big]\Big\} \geq Ew(\zeta),$$

where $l_0 = (A_0, \alpha_0, \psi_0)'$ has positive components. This implies Theorem 6.

6 Conclusion

We considered the MLE in the semi-parametric Nevzorov's record model and in the three-parametric Nevzorov's model, which is based on Fréchet distribution. Basing on LAN property, we proved that both estimators are efficient in the sense of Hajék bound. This result is not a consequence of general theory of efficiency presented in Ibragimov and Has'minskii (1981), because our considerations are not identically distributed. The efficiency result is important in insurance business. It shows that the maximum likelihood approach yields the asymptotically optimal way to predict future claims, arising out of natural catastrophes.

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