

Hypothesis testing of the drift parameter sign for fractional Ornstein–Uhlenbeck process

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Abstract: We consider the fractional Ornstein–Uhlenbeck process with an unknown drift parameter and known Hurst parameter H . We propose a new method to test the hypothesis of the sign of the parameter and prove the consistency of the test. Contrary to the previous works, our approach is applicable for all $H \in (0, 1)$.

MSC 2010 subject classifications: 60G22, 62F03, 62F05.

Keywords and phrases: Fractional Brownian motion, fractional Ornstein–Uhlenbeck process, hypothesis testing.

Received September 2016.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space, and $B^H = \{B_t^H, t \in \mathbb{R}\}$ be a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ on this probability space, that is a centered Gaussian process with covariance function

$$\mathbf{E}B_t^H B_s^H = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t - s|^{2H} \right), \quad t, s \in \mathbb{R}.$$

Since $\mathbf{E}(B_t^H - B_s^H)^2 = |t - s|^{2H}$ and the process B^H is Gaussian, it has a continuous modification by Kolmogorov’s theorem. In what follows we consider such modification.

The present paper deals with the inference problem associated with the Langevin equation

$$X_t = x_0 + \theta \int_0^t X_s ds + B_t^H, \quad t \geq 0, \quad (1)$$

where $x_0 \in \mathbb{R}$, and $\theta \in \mathbb{R}$ is an unknown drift parameter. This equation has a unique solution, see [5]. In order to avoid integration with respect to the fractional Brownian motion for $0 < H < 1/2$, we can write this solution in the following form

$$X_t = x_0 e^{\theta t} + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H ds + B_t^H, \quad t \geq 0. \quad (2)$$

The process $X = \{X_t, t \geq 0\}$ is called a *fractional Ornstein–Uhlenbeck process* [5]. It is a Gaussian process, consequently its one-dimensional distributions are normal, with mean $x_0 e^{\theta t}$ and variance

$$v(\theta, t) = H \int_0^t s^{2H-1} (e^{\theta s} + e^{\theta(2t-s)}) ds,$$

see Lemma A.1 in Appendix.

The estimation problem for the drift parameter θ in the model (1) was studied in many works. We refer to the paper [15] for the extended survey of these results. The maximum likelihood estimators (MLE's) were studied in [13, 18, 19] for $H \geq 1/2$ and in the paper [20] for $H < 1/2$. Note that the MLE is hardly discretized because it contains the stochastic integrals with singular kernels. Therefore, several nonstandard estimators have been proposed recently. In particular, for $\theta < 0$ (the ergodic case) and $H \geq 1/2$ Hu and Nualart [11] constructed the analog of the least-squares estimator of the form

$$\hat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt}, \quad (3)$$

where the integral $\int_0^T X_t dX_t$ is the divergence-type one. As an alternative, they considered the estimator

$$\hat{\theta}_T = - \left(\frac{1}{H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}. \quad (4)$$

In the non-ergodic case, when $\theta > 0$, Belfadli et al. [1] proposed the estimator

$$\hat{\theta}_T = \frac{X_T^2}{2 \int_0^T X_t^2 dt} \quad (5)$$

and proved its strong consistency for $H > 1/2$. Later in [6] its strong consistency was obtained for all $H \in (0, 1)$. Note that the estimator (5) coincides with the

estimator (3), where the integral $\int_0^T X_t dX_t$ is understood in the path-wise sense. In the papers [3, 4, 7, 8, 9, 12, 15, 21] the discretized estimators were considered. For the case $\theta < 0$ the drift parameter estimator based on polynomial variations was proposed in [10].

The methods of constructing the estimators and their asymptotic properties essentially depend on the sign of unknown drift parameter θ . In particular, the estimator (4) is based on the ergodicity and does not work in the non-ergodic case. Similarly, the estimator (5) converges to zero if $\theta < 0$, see remark at the end of Sec. 3 in [11].

The above discussion motivates the hypothesis testing of the sign of drift parameter in the model (1). The interest to this problem is also connected with the stability properties of the solution to the equation (1), which also depend on the sign of θ . For $H \geq 1/2$, this problem was studied by Moers [16]. He constructed a test using the estimator

$$\tilde{\theta}_{T,H} = \frac{X_T^2 - X_0^2}{2 \int_0^T X_t^2 dt} - \left(\frac{1}{H\Gamma(2H)T} \int_0^T X_t^2 dt \right)^{-\frac{1}{2H}}. \quad (6)$$

The exact distribution of $\tilde{\theta}_{T,H}$ is not known, and the test is based on the asymptotic distribution of $T\tilde{\theta}_{T,H}$. The values of the corresponding test statistic should be compared with quantiles of the random variable

$$\frac{(B_1^H)^2}{2 \int_0^1 (B_t^H)^2 dt} - \left(\frac{1}{H\Gamma(2H)} \int_0^1 (B_t^H)^2 dt \right)^{-\frac{1}{2H}}, \quad (7)$$

and the quantiles can be obtained by Monte Carlo simulation. The test can be used for testing three types of hypothesis: $H_0: \theta \geq 0$ against $H_1: \theta < 0$, $H_0: \theta \leq 0$ against $H_1: \theta > 0$, and $H_0: \theta = 0$ against $H_1: \theta \neq 0$. The consistency of the test is proved only for $H \in [1/2, 3/4)$ for a simple alternative $\theta_1 < 0$, and for $H \in [1/2, 1)$ for $\theta_1 > 0$. Tanaka [18, 19] considered the testing of the hypothesis $H_0: \theta = 0$ against the alternatives $H_1: \theta < 0$ and $H_1: \theta > 0$. He proposed tests based on the MLE (for both alternatives) and on the minimum contrast estimator (only for the ergodic case). Those tests were considered also for $H \geq 1/2$. To the best of our knowledge, there are no tests, suitable for the discrete-time observations of the process.

In the present paper we propose comparatively a simple test for testing the null hypothesis $H_0: \theta \leq 0$ against the alternative $H_1: \theta > 0$. The main advantage of our approach is that it can be used for any $H \in (0, 1)$. The distribution of the test statistic is computed explicitly, and the power of test can be found numerically for any given simple alternative. Also we consider the hypothesis testing $H_0: \theta \geq \theta_0$ against $H_1: \theta \leq 0$, where $\theta_0 \in (0, 1)$ is some fixed number. Unfortunately, our approach does not enable to test the hypothesis $H_0: \theta = 0$ against the two-sided alternative $H_1: \theta \neq 0$. The test is based on the observations of the process X at two points: 0 and T . Therefore, it is applicable for both continuous and discrete cases.

Remark 1.1. The situation is similar to the model $Y_t = at + W_t$, where W_t is a Wiener process. In this case the MLE of the drift parameter by observations at points $0 = t_0 \leq t_1 \leq \dots \leq t_N = T$ is given by

$$\hat{a} = \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} (Y_{t_{i+1}} - Y_{t_i}) = \frac{Y_T - Y_0}{T},$$

and depends on the observations at two points, see e. g. [2, p. 363]. Similarly, in Samuelson's model [17] of the form $\log S_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t$ with constant drift parameter μ and known volatility σ , the MLE of μ equals

$$\hat{\mu} = \frac{\log S_T - \log S_0}{T} + \frac{\sigma^2}{2}.$$

and also depends on the observations at two points.

The paper is organized as follows. In Section 2 the problem of hypothesis testing for the sign of the drift parameter θ is considered. Section 3 is devoted to numerics. In Appendix we get some auxiliary results. In particular, we calculate the first two moments of the fractional Ornstein–Uhlenbeck process.

2. Hypothesis testing of the drift parameter sign

2.1. Test statistic

For hypothesis testing of the sign of the parameter θ we construct a test based on the asymptotic behavior of the random variable

$$Z(t) = \frac{\log^+ \log |X_t|}{\log t}, \quad t > 1, \quad (8)$$

where $\log^+ x = \log x$ when $x > 1$ and $\log^+ x = 0$ otherwise. The following result explains the main idea. It is based on the different asymptotic behavior of the fractional Ornstein–Uhlenbeck process with positive drift parameter and negative one.

Lemma 2.1. *The value of $Z(t)$ converges a. s. to 1 for $\theta > 0$, and to 0 for $\theta \leq 0$, as $t \rightarrow \infty$.*

Proof. For $\theta > 0$, by [6, Lemma 2.1] (see also [1] for $H > 1/2$),

$$e^{-\theta t} X_t \rightarrow \xi_\theta \quad \text{a. s., as } t \rightarrow \infty,$$

where $\xi_\theta = x_0 + \theta \int_0^\infty e^{-\theta s} B_s^H ds \simeq \mathcal{N}\left(x_0, \frac{H\Gamma(2H)}{\theta^{2H}}\right)$. Then

$$\log |X_t| - \theta t \rightarrow \log |\xi_\theta| \quad \text{a. s., as } t \rightarrow \infty,$$

where ξ_θ is a normal random variable, hence, $0 < |\xi_\theta| < \infty$ a. s. Therefore,

$$\frac{\log |X_t|}{t} \rightarrow \theta \quad \text{a. s., as } t \rightarrow \infty.$$

It means that there exists $\Omega' \subset \Omega$ such that $\mathbf{P}(\Omega') = 1$ and for any $\omega \in \Omega'$ there exists $t(\omega)$ such that for $t \geq t(\omega)$: $\log |X_t| > 0$. Hence, for $t \geq t(\omega)$ we have that

$$\begin{aligned} \left| \frac{\log^+ \log |X_t|}{\log t} - 1 \right| &= \left| \frac{\log \log |X_t|}{\log t} - 1 \right| \\ &= \left| \frac{\log \log |X_t| - \log t}{\log t} \right| = \left| \frac{\log \frac{\log |X_t|}{t}}{\log t} \right| \rightarrow 0 \end{aligned}$$

a. s., as $t \rightarrow \infty$. For $\theta \leq 0$, it follows from (19) that

$$\begin{aligned} |Z(t)| &\leq \left| \frac{\log^+ (\log (1 + t^H \log^2 t) + \log \zeta)}{\log t} \right| \\ &= \left| \frac{\log (\log (1 + t^H \log^2 t) + \log \zeta)}{\log t} \right| \sim \left| \frac{\log (\log (t^H \log^2 t))}{\log t} \right| \rightarrow 0, \end{aligned}$$

as $t \rightarrow \infty$. □

The next result gives the cdf of $Z(t)$. Let Φ and φ denote the cdf and pdf, respectively, of the standard normal variable.

Lemma 2.2. *For $t > 1$ the probability $g(\theta, x_0, t, c) = \mathbf{P}(Z(t) \leq c)$ is given by*

$$g(\theta, x_0, t, c) = \Phi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) + \Phi \left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) - 1, \tag{9}$$

and g is a decreasing function of $\theta \in \mathbb{R}$.

Proof. Using Lemma A.1 and taking into account that $\log^+ x$ is a non-decreasing function, we obtain

$$\begin{aligned} \mathbf{P}(Z(t) \leq c) &= \mathbf{P}(|X_t| \leq e^{t^c}) = \Phi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) - \Phi \left(\frac{-e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) \\ &= \Phi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) + \Phi \left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) - 1. \end{aligned}$$

Let us prove the monotonicity of the function g with respect to θ . Note that g is an even function with respect to x_0 . Therefore, it suffices to consider only the

case $x_0 \geq 0$. The partial derivative equals

$$\begin{aligned} \frac{\partial g}{\partial \theta} &= \varphi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) \left(-x_0 t e^{\theta t} v^{-\frac{1}{2}}(\theta, t) - \frac{1}{2} v^{-\frac{3}{2}}(\theta, t) v'_\theta(\theta, t) (e^{t^c} - x_0 e^{\theta t}) \right) \\ &\quad + \varphi \left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) \left(x_0 t e^{\theta t} v^{-\frac{1}{2}}(\theta, t) - \frac{1}{2} v^{-\frac{3}{2}}(\theta, t) v'_\theta(\theta, t) (e^{t^c} + x_0 e^{\theta t}) \right) \\ &= -\frac{1}{2} v^{-\frac{3}{2}}(\theta, t) v'_\theta(\theta, t) e^{t^c} \left(\varphi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) + \varphi \left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) \right) \\ &\quad - x_0 e^{\theta t} v^{-\frac{3}{2}}(\theta, t) \left(t v(\theta, t) - \frac{1}{2} v'_\theta(\theta, t) \right) \\ &\quad \times \left(\varphi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) - \varphi \left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) \right). \end{aligned} \tag{10}$$

Since

$$v'_\theta(\theta, t) = H \int_0^t s^{2H-1} (s e^{\theta s} + (2t-s) e^{\theta(2t-s)}) ds > 0, \tag{11}$$

we see that the first term in the right-hand side of (10) is negative. Let us consider the second term. From (16) and (11) it follows that

$$t v(\theta, t) - \frac{1}{2} v'_\theta(\theta, t) = H \int_0^t s^{2H-1} \left((t - \frac{1}{2}s) e^{\theta s} + \frac{1}{2} s e^{\theta(2t-s)} \right) ds > 0.$$

Since $|e^{t^c} - x_0 e^{\theta t}| \leq e^{t^c} + x_0 e^{\theta t}$ for $x_0 \geq 0$, we have

$$\varphi \left(\frac{e^{t^c} - x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) - \varphi \left(\frac{e^{t^c} + x_0 e^{\theta t}}{\sqrt{v(\theta, t)}} \right) \geq 0.$$

Thus, the second term in the right-hand side of (10) is non-positive. Hence, $\frac{\partial g}{\partial \theta} < 0$. \square

2.2. Testing the hypothesis $H_0: \theta \leq 0$ against $H_1: \theta > 0$

To test $H_0: \theta \leq 0$ against the alternative $H_1: \theta > 0$, we consider the following procedure: for a given significance level α , and for sufficiently large value of t we choose a threshold $c = c_t \in (0, 1)$, see Lemma 2.3. Further, when $Z(t) \leq c$ the hypothesis H_0 cannot be rejected, and when $Z(t) > c$ it is rejected. Below we will propose a technically simpler version of this test, without the computation of c , see Remark 2.4. The threshold c can be chosen as follows.

Fix a number $\alpha \in (0, 1)$, the significance level of the test. This level gives the maximal probability of a type I error, that is in our case the probability to

reject the hypothesis $H_0: \theta \leq 0$ when it is true. By Lemma 2.2, for a threshold $c \in (0, 1)$ and $t > 1$ this probability equals

$$\sup_{\theta \leq 0} \mathbf{P}(Z(t) > c) = 1 - g(0, x_0, t, c).$$

Therefore, we determine c_t as a solution to the equation

$$g(0, x_0, t, c_t) = 1 - \alpha. \quad (12)$$

The following result shows that for any $\alpha \in (0, 1)$, it is possible to choose a sufficiently large t such that $c_t \in (0, 1)$.

Lemma 2.3. *Let $\alpha \in (0, 1)$. Then there exists $t_0 \geq 1$ such that for all $t > t_0$ there exists a unique $c_t \in (0, 1)$ such that $g(0, x_0, t, c_t) = 1 - \alpha$. Moreover $c_t \rightarrow 0$, as $t \rightarrow \infty$.*

The constant t_0 can be chosen as the largest $t \geq 1$ that satisfies at least one of the following two equalities

$$g(0, x_0, t, 0) = 1 - \alpha \quad \text{or} \quad g(0, x_0, t, 1) = 1 - \alpha. \quad (13)$$

Proof. By Lemma A.3 (iii), $v(0, t) = t^{2H}$. Then for $\theta = 0$ the formula (9) becomes

$$g(0, x_0, t, c) = \Phi\left(\frac{e^{tc} - x_0}{t^H}\right) + \Phi\left(\frac{e^{tc} + x_0}{t^H}\right) - 1. \quad (14)$$

For any $t > 1$, the function $g(0, x_0, t, c)$ is strictly increasing with respect to c . For $c = 0$ we have

$$g(0, x_0, t, 0) = \Phi\left(\frac{e - x_0}{t^H}\right) + \Phi\left(\frac{e + x_0}{t^H}\right) - 1 \rightarrow 2\Phi(0) - 1 = 0, \quad \text{as } t \rightarrow \infty.$$

Therefore, there exists $t_1 > 1$ such that $g(0, x_0, t, 0) < 1 - \alpha$ for all $t \geq t_1$.

Similarly, for $c = 1$

$$g(0, x_0, t, 1) = \Phi\left(\frac{e^t - x_0}{t^H}\right) + \Phi\left(\frac{e^t + x_0}{t^H}\right) - 1 \rightarrow 2\Phi(\infty) - 1 = 1, \quad \text{as } t \rightarrow \infty.$$

Therefore, there exists $t_2 > 1$ such that $g(0, x_0, t, 1) > 1 - \alpha$ for all $t \geq t_2$.

Thus, for any $t \geq t_0 = \max\{t_1, t_2\}$ there exists a unique $c_t \in (0, 1)$ such that $g(0, x_0, t, c_t) = 1 - \alpha$.

To prove the convergence $c_t \rightarrow 0$, $t \rightarrow \infty$, consider an arbitrary $\varepsilon \in (0, 1)$. Then

$$g(0, x_0, t, \varepsilon) = \Phi\left(\frac{e^{t\varepsilon} - x_0}{t^H}\right) + \Phi\left(\frac{e^{t\varepsilon} + x_0}{t^H}\right) - 1 \rightarrow 2\Phi(\infty) - 1 = 1, \quad \text{as } t \rightarrow \infty.$$

Arguing as above, we see that there exists $t_3 > 1$ such that for any $t > t_3$ the unique $c_t \in (0, 1)$, for which $g(0, x_0, t, c_t) = 1 - \alpha$, belongs to the interval $(0, \varepsilon)$. This implies the convergence $c_t \rightarrow 0$, as $t \rightarrow \infty$.

It follows from (14) that $g(0, x_0, t, 0) = g(0, x_0, t, 1)$ for $t = 1$. As $t \rightarrow \infty$, we have $g(0, x_0, t, 0) \rightarrow 0$, $g(0, x_0, t, 1) \rightarrow 1$. Hence, at least one of the equalities (13) is satisfied for some $t \geq 1$ and the set of such t 's is bounded. \square

Remark 2.4. Since the function $g(0, x_0, t, c)$ is strictly increasing with respect to c for $t > 1$, we see that the inequality $Z(t) \leq c_t$ is equivalent to the inequality $g(0, x_0, t, Z(t)) \leq g(0, x_0, t, c_t) = 1 - \alpha$. Therefore, we do not need to compute the value of c_t . It is sufficient to compare $g(0, x_0, t, Z(t))$ with the level $1 - \alpha$.

Algorithm 2.5. The hypothesis $H_0: \theta \leq 0$ against the alternative $H_1: \theta > 0$ can be tested as follows.

1. Choose $0 < \alpha < 1$. Find t_0 defined in Lemma 2.3. The algorithm can be applied only in the case $t > t_0$.
2. Evaluate the statistic $Z(t)$ defined by (8).
3. Compute the value of $g(0, x_0, t, Z(t))$, see (14).
4. Do not reject the hypothesis H_0 if $g(0, x_0, t, Z(t)) \leq 1 - \alpha$, and reject it otherwise.

Remark 2.6. In fact, the condition $t > t_0$ is not too restrictive, since for reasonable values of α , the values of t_0 are quite small, see Table 1.

Let us summarize the properties of the test in the following theorem.

Theorem 2.7. The test described in Algorithm 2.5 is unbiased and consistent, as $t \rightarrow \infty$. For a simple alternative $\theta_1 > 0$ and moment $t > t_0$, the power of the test equals $1 - g(\theta_1, x_0, t, c_t)$, where c_t can be found from (12).

Proof. It follows from the monotonicity of g with respect to θ (see Lemma 2.2) that for any $\theta_1 > 0$

$$\mathbf{P}(Z(t) > c_t) = 1 - g(\theta_1, x_0, t, c_t) > 1 - g(0, x_0, t, c_t) = \alpha.$$

This means that the test is unbiased. Evidently, for a simple alternative $\theta_1 > 0$ the power of the test equals $1 - g(\theta_1, x_0, t, c_t)$.

It follows from the convergence $c_t \rightarrow 0$, as $t \rightarrow \infty$ (see Lemma 2.3), that $c_t < c$ for sufficiently large t and some constant $c \in (0, 1)$. Taking into account the formula (9) and Lemma A.3 (i), we get, as $t \rightarrow \infty$:

$$\begin{aligned} 1 &\geq 1 - g(\theta_1, x_0, t, c_t) \geq 1 - g(\theta_1, x_0, t, c) \\ &= 2 - \Phi\left(\frac{e^{t^c} - x_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) - \Phi\left(\frac{e^{t^c} + x_0 e^{\theta_1 t}}{\sqrt{v(\theta_1, t)}}\right) \\ &\rightarrow 2 - \Phi\left(-\frac{x_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) - \Phi\left(\frac{x_0 \theta_1^H}{\sqrt{H\Gamma(2H)}}\right) = 1. \end{aligned}$$

Hence, the test is consistent. □

Remark 2.8. For the composite alternative $H_1: \theta > 0$, the power of the test of Algorithm 2.5 is small and equals

$$\inf_{\theta_1 > 0} (1 - g(\theta_1, x_0, t, c_t)) = 1 - g(0, x_0, t, c_t) = \alpha.$$

We can consider testing of $H_0: \theta \leq 0$ against the alternative $H_1: \theta \geq \theta_0$, where $\theta_0 > 0$ is a fixed number. For this composite alternative the power of the test of Algorithm 2.5 equals

$$\inf_{\theta_1 \geq \theta_0} (1 - g(\theta_1, x_0, t, c_t)) = 1 - g(\theta_0, x_0, t, c_t)$$

and tends to 1, as $t \rightarrow \infty$.

2.3. Testing the hypothesis $H_0: \theta \geq \theta_0$ against $H_1: \theta \leq 0$

Fix $\theta_0 \in (0, 1)$. Consider the problem of testing the hypothesis $H_0: \theta \geq \theta_0$ against alternative $H_1: \theta \leq 0$.

Algorithm 2.9. *The hypothesis $H_0: \theta \geq \theta_0$ against the alternative $H_1: \theta \leq 0$ can be tested as follows.*

1. Choose $0 < \alpha < 1$. Find \tilde{t}_0 defined in Lemma 2.10. The algorithm can be applied only in the case $t > \tilde{t}_0$.
2. Evaluate the statistic $Z(t)$ defined by (8).
3. Compute the value of $g(\theta_0, x_0, t, Z(t))$, see (9).
4. Do not reject the hypothesis H_0 if $g(\theta_0, x_0, t, Z(t)) \geq \alpha$, and reject it otherwise.

This algorithm is based on the following results. They can be proved similarly to the previous subsection.

Lemma 2.10. *Let $\alpha \in (0, 1)$. There exists $\tilde{t}_0 > 1$ such that for all $t > \tilde{t}_0$ there exists a unique $\tilde{c}_t \in (0, 1)$ such that*

$$g(\theta_0, x_0, t, \tilde{c}_t) = \alpha. \quad (15)$$

In this case $\tilde{c}_t \rightarrow 1$, as $t \rightarrow \infty$.

The constant \tilde{t}_0 can be chosen as the largest $t > 1$ that satisfies at least one of the following two equalities

$$g(\theta_0, x_0, t, 0) = \alpha \quad \text{or} \quad g(\theta_0, x_0, t, 1) = \alpha.$$

Theorem 2.11. *The test described in Algorithm 2.9 is unbiased and consistent, as $t \rightarrow \infty$. For a simple alternative $\theta_1 \leq 0$ and moment $t > \tilde{t}_0$, the power of the test equals $g(\theta_1, x_0, t, \tilde{c}_t)$, where \tilde{c}_t can be found from (15). For the composite alternative $H_1: \theta \leq 0$, the power of the test equals $g(0, x_0, t, \tilde{c}_t)$ and tends to 1, as $t \rightarrow \infty$.*

Remark 2.12. The values of \tilde{t}_0 for various values of θ_0 and H are represented in Table 2. We see that if θ_0 is too close to zero, then for small H , the condition $t > \tilde{t}_0$ does not hold for reasonable values of t .

Remark 2.13. If we have a confidence interval for θ , then the value of $\theta_0 \in (0, 1)$ can be chosen less than or equal to a lower confidence bound (in the case when the latter is positive).

TABLE 1
Value of t_0 for various H and α ($x_0 = 1$)

H	$\alpha = 0.01$	$\alpha = 0.05$
0.1	1.2157	1.5310
0.2	1.2313	1.2373
0.3	1.2492	1.1526
0.4	1.2699	1.1124
0.5	1.2940	1.0889
0.6	1.3224	1.0736
0.7	1.3561	1.0627
0.8	1.3968	1.0547
0.9	1.4462	1.0485

TABLE 2
Values of \tilde{t}_0 for various H and θ_0 ($x_0 = 1$, $\alpha = 0.05$)

H	$\theta_0 = 0.1$	$\theta_0 = 0.05$	$\theta_0 = 0.01$	$\theta_0 = 0.001$	$\theta_0 = 0$
0.1	32.433	65.242	326.47	3193.6	$2.3369 \cdot 10^{16}$
0.2	32.667	64.721	307.43	2719.1	$1.5287 \cdot 10^8$
0.3	31.994	61.728	271.64	2073.5	285 900
0.4	30.592	57.078	227.99	1387.8	12 364
0.5	28.659	51.413	181.64	778.94	1878.1
0.6	26.375	45.233	137.06	382.06	534.70
0.7	23.903	38.967	98.759	189.71	217.96
0.8	21.386	32.995	69.618	104.11	111.19
0.9	18.946	27.621	49.408	63.576	65.878

3. Numerical illustrations

In this section we illustrate the performance of our algorithms by simulation experiments. We choose the initial value $x_0 = 1$ for all simulations.

In Tables 1–2 the values of t_0 and \tilde{t}_0 for various H and θ_0 are given.

We simulate fractional Brownian motion at the points $t = 0, h, 2h, 3h, \dots$ and compute the approximate values of the Ornstein–Uhlenbeck process as the solution to the equation (1), using Euler’s approximations. For various values of θ we simulate $n = 1000$ sample paths with the step $h = 1/10000$. Then we apply our algorithms, choosing the significance level $\alpha = 0.05$. In Table 3 the empirical rejection probabilities of the test of Algorithm 2.5 for the hypothesis testing $H_0: \theta \leq 0$ against the alternative $H_1: \theta > 0$ for $H = 0.3$ and $H = 0.7$ are reported.

Then we test the same hypothesis using the test of Moers [16] for $H = 0.7$. By Monte Carlo simulations for 20 000 sample paths of the process $\{B_t^H, t \in [0, 1]\}$ we estimate the $(1 - \alpha)$ -quantile $\psi_{1-\alpha}$ of the distribution (7) for $\alpha = 0.05$. Then we compare the statistic $T\tilde{\theta}_{T,H}$ (see (6)) with the value of this quantile and reject the hypothesis $H_0: \theta \leq 0$ if $T\tilde{\theta}_{T,H} > \psi_{1-\alpha}$. We obtained that $\psi_{0.95} \approx 0.827946$. The empirical rejection probabilities for this test are given in Table 4. We see

TABLE 3
 Empirical rejection probabilities of the test of Algorithm 2.5 for the hypothesis testing $H_0: \theta \leq 0$ against the alternative $H_1: \theta > 0$ for $H = 0.3$ and $H = 0.7$

θ	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3
$H = 0.3$									
$T = 20$	0.000	0.003	0.043	0.341	0.701	0.880	0.973	0.982	0.996
$T = 40$	0.000	0.000	0.043	0.675	0.952	0.995	0.999	1.000	1.000
$T = 60$	0.000	0.000	0.039	0.860	0.994	1.000	1.000	1.000	1.000
$T = 80$	0.000	0.000	0.048	0.940	1.000	1.000	1.000	1.000	1.000
$T = 100$	0.000	0.000	0.049	0.986	1.000	1.000	1.000	1.000	1.000
$H = 0.7$									
$T = 20$	0.000	0.001	0.058	0.284	0.540	0.800	0.910	0.967	0.979
$T = 40$	0.000	0.000	0.050	0.581	0.889	0.984	0.998	1.000	1.000
$T = 60$	0.000	0.000	0.042	0.782	0.980	1.000	0.999	1.000	1.000
$T = 80$	0.000	0.000	0.047	0.908	0.995	1.000	1.000	1.000	1.000
$T = 100$	0.000	0.000	0.048	0.959	1.000	1.000	1.000	1.000	1.000

TABLE 4
 Empirical rejection probabilities of the test of Moers [16] for the hypothesis testing $H_0: \theta \leq 0$ against the alternative $H_1: \theta > 0$ for $H = 0.7$

θ	-0.1	-0.05	0	0.05	0.1	0.15	0.2	0.25	0.3
$T = 20$	0.001	0.013	0.085	0.370	0.706	0.873	0.947	0.976	0.992
$T = 40$	0.000	0.004	0.095	0.682	0.948	0.993	0.999	1.000	1.000
$T = 60$	0.000	0.002	0.092	0.881	0.995	1.000	1.000	1.000	1.000
$T = 80$	0.000	0.000	0.105	0.948	0.999	1.000	1.000	1.000	1.000
$T = 100$	0.000	0.000	0.089	0.977	1.000	1.000	1.000	1.000	1.000

that comparing to our algorithm, the test of Moers has bigger power, i.e., it works a bit better when the alternative is true. But for $\theta = 0$, the necessary significance level $\alpha = 0.05$ is not achieved.

TABLE 5
 Empirical rejection probabilities of the test of Algorithm 2.9 for the hypothesis testing $H_0: \theta \geq \theta_0$ against the alternative $H_1: \theta \leq 0$ for $\theta_0 = 0.1$, $H = 0.3$ and $H = 0.7$

θ	-0.25	-0.2	-0.15	-0.1	-0.05	0	0.1	0.15	0.2
$H = 0.3$									
$T = 40$	1.000	1.000	1.000	1.000	1.000	0.938	0.056	0.008	0.000
$T = 60$	1.000	1.000	1.000	1.000	1.000	1.000	0.052	0.005	0.000
$T = 80$	1.000	1.000	1.000	1.000	1.000	1.000	0.054	0.001	0.000
$T = 100$	1.000	1.000	1.000	1.000	1.000	1.000	0.054	0.001	0.000
$H = 0.7$									
$T = 40$	1.000	1.000	1.000	1.000	0.978	0.689	0.051	0.006	0.001
$T = 60$	1.000	1.000	1.000	1.000	1.000	1.000	0.052	0.004	0.001
$T = 80$	1.000	1.000	1.000	1.000	1.000	1.000	0.051	0.003	0.000
$T = 100$	1.000	1.000	1.000	1.000	1.000	1.000	0.051	0.001	0.000

TABLE 6
 Empirical rejection probabilities of the test of Algorithm 2.9 for the hypothesis testing
 $H_0: \theta \geq \theta_0$ against the alternative $H_1: \theta \leq 0$ for $\theta_0 = 0.05$, $H = 0.7$

θ	-0.25	-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15
$T = 40$	0.842	0.773	0.661	0.566	0.368	0.149	0.051	0.007	0.000
$T = 60$	0.999	1.000	0.990	0.955	0.799	0.346	0.047	0.002	0.000
$T = 80$	1.000	1.000	1.000	1.000	0.999	0.719	0.044	0.001	0.000
$T = 100$	1.000	1.000	1.000	1.000	1.000	0.980	0.050	0.000	0.000

Tables 5 and 6 represent empirical rejection probabilities of the test of Algorithm 2.9 for $\theta_0 = 0.1$ and $\theta_0 = 0.05$, respectively. We see that the test power increases if θ_0 increases. Also, the test power tends to 1, as the time horizon T increases. Hence these simulation studies confirm the theoretical results on the consistency.

Appendix A

A.1. One-dimensional distribution of the fractional Ornstein–Uhlenbeck process

Let $\{X_t, t \geq 1\}$ be the fractional Ornstein–Uhlenbeck process defined by (1).

Lemma A.1. *The random variable X_t has normal distribution $\mathcal{N}(x_0 e^{\theta t}, v(\theta, t))$, with variance*

$$v(\theta, t) = H \int_0^t s^{2H-1} \left(e^{\theta s} + e^{\theta(2t-s)} \right) ds. \quad (16)$$

Proof. Since B_t^H is a centered Gaussian process, it immediately follows from (2) that X_t has normal distribution with mean $x_0 e^{\theta t}$. Let us calculate its variance. We have

$$\begin{aligned} \text{Var } X_t &= \mathbf{E} \left(B_t^H + \theta e^{\theta t} \int_0^t e^{-\theta s} B_s^H ds \right)^2 \\ &= \mathbf{E} \left[(B_t^H)^2 \right] + 2\theta e^{\theta t} \int_0^t e^{-\theta s} \mathbf{E} [B_t^H B_s^H] ds \\ &\quad + \theta^2 e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} \mathbf{E} [B_s^H B_u^H] ds du \\ &= t^{2H} + \theta e^{\theta t} \int_0^t e^{-\theta s} (t^{2H} + s^{2H} - (t-s)^{2H}) ds \\ &\quad + \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} (s^{2H} + u^{2H} - |s-u|^{2H}) ds du \\ &= t^{2H} + e^{\theta t} t^{2H} (1 - e^{-\theta t}) + \theta e^{\theta t} \int_0^t e^{-\theta s} s^{2H} ds \end{aligned}$$

$$\begin{aligned}
& -\theta e^{\theta t} \int_0^t e^{-\theta(t-u)} u^{2H} du + \theta^2 e^{2\theta t} \int_0^t e^{-\theta s} s^{2H} ds \int_0^t e^{-\theta u} du \quad (17) \\
& -\frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s - u|^{2H} ds du \\
& = e^{\theta t} t^{2H} + \theta e^{\theta t} \int_0^t e^{-\theta s} s^{2H} ds - \theta \int_0^t e^{\theta s} s^{2H} ds \\
& \quad + \theta e^{2\theta t} (1 - e^{-\theta t}) \int_0^t e^{-\theta s} s^{2H} ds \\
& \quad - \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s - u|^{2H} ds du \\
& = e^{\theta t} t^{2H} - \theta \int_0^t e^{\theta s} s^{2H} ds + \theta e^{2\theta t} \int_0^t e^{-\theta s} s^{2H} ds \\
& \quad - \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s - u|^{2H} ds du.
\end{aligned}$$

The last summand can be rewritten as follows

$$\begin{aligned}
& \frac{\theta^2}{2} e^{2\theta t} \int_0^t \int_0^t e^{-\theta s - \theta u} |s - u|^{2H} ds du \\
& = \frac{\theta^2}{2} e^{2\theta t} \left(\int_0^t \int_0^s e^{-\theta s - \theta u} (s - u)^{2H} du ds + \int_0^t \int_s^t e^{-\theta s - \theta u} (u - s)^{2H} du ds \right) \\
& = \theta^2 e^{2\theta t} \int_0^t \int_0^s e^{-\theta s - \theta u} (s - u)^{2H} du ds = \theta^2 e^{2\theta t} \int_0^t \int_0^s e^{-2\theta s + \theta v} v^{2H} dv ds \\
& = \frac{\theta}{2} e^{2\theta t} \int_0^t e^{-\theta v} v^{2H} dv - \frac{\theta}{2} \int_0^t e^{\theta v} v^{2H} dv.
\end{aligned}$$

Substituting the latter value into the above formula (17) for $\text{Var } X_t$, we get

$$\begin{aligned}
\text{Var } X_t & = e^{\theta t} t^{2H} - \frac{\theta}{2} \int_0^t e^{\theta s} s^{2H} ds + \frac{\theta}{2} e^{2\theta t} \int_0^t e^{-\theta s} s^{2H} ds \\
& = e^{\theta t} t^{2H} - \frac{1}{2} \int_0^t s^{2H} d(e^{\theta s} + e^{2\theta t - \theta s}) \\
& = H \int_0^t s^{2H-1} (e^{\theta s} + e^{2\theta t - \theta s}) ds. \quad \square
\end{aligned}$$

Remark A.2. The variance of the fractional Ornstein–Uhlenbeck process was also computed in [6]. Here we derive a quite simple formula for it, which is more suitable for our purposes.

As a direct corollary from Lemma A.1, we get the asymptotical behavior of the function $v(\theta, t)$, as $t \rightarrow \infty$.

Lemma A.3. (i) If $\theta > 0$, then $v(\theta, t) \sim \frac{H\Gamma(2H)}{\theta^{2H}} e^{2\theta t}$, as $t \rightarrow \infty$.
(ii) If $\theta < 0$, then $v(\theta, t) \rightarrow \frac{H\Gamma(2H)}{(-\theta)^{2H}}$, as $t \rightarrow \infty$.

(iii) $v(0, t) = t^{2H}$, $t \geq 0$.

Proof. (i) If $\theta > 0$, then by formula (16),

$$\frac{v(\theta, t)}{e^{2\theta t}} = H e^{-2\theta t} \int_0^t s^{2H-1} e^{\theta s} ds + H \int_0^t s^{2H-1} e^{-\theta s} ds \rightarrow \frac{H\Gamma(2H)}{\theta^{2H}}, \text{ as } t \rightarrow \infty.$$

(ii) Note that $v(\theta, t) = e^{2\theta t} v(-\theta, t)$, by (16). Then the convergence follows from (i).

(iii) The statement follows directly from (16). \square

Remark A.4. The statement (i) was proved in [11] for the case $H \in [1/2, 1)$ and in [6] for all $H \in (0, 1)$. The statement (ii) for $H > 1/2$ was obtained in [1].

A.2. Almost sure bounds for the fractional Ornstein–Uhlenbeck process

Lemma A.5 ([14, 15]). *There exists a nonnegative random variable ζ such that for all $s > 0$, the following inequalities hold true:*

$$\sup_{0 \leq s \leq t} |B_s^H| \leq (1 + t^H \log^2 t) \zeta, \quad (18)$$

and for $\theta \leq 0$

$$\sup_{0 \leq s \leq t} |X_s| \leq (1 + t^H \log^2 t) \zeta. \quad (19)$$

Moreover, ζ has the following property: there exists $C > 0$ such that $\mathbf{E} \exp\{x\zeta^2\} < \infty$, for any $0 < x < C$.

Proof. The bound (18) was established in [14], see also Eq. (14) in [15]. It also implies that the inequality (19) holds for $\theta = 0$, since in this case $X_t = x_0 + B_t^H$. The bound (19) for $\theta < 0$ was obtained in [15, Eq. (19)]. \square

Acknowledgement

The authors are grateful to the anonymous referees for careful reading of the paper and helpful remarks.

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