Theory of Stochastic Processes
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Goodness-of-fit tests in Nevzorov's model

Goodness-of-fit test is studied for Nevzorov's model.

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1. INTRODUCTION

We want to present an approach to the investigation of natural catastrophe claims in the presence of a trend. The so-called *Nevzorov's model* is studied. We assume that the yearly catastrophe claims are realizations of an independent sequence $\{X_i, i \geq 1\}$ of random variables with support $R^+ := [0, \infty)$ and continuous cumulative d.f. $\{F_i, i \geq 1\}$, s.t. $F_i = F^{\gamma_i}$, with $\gamma_i = \gamma^{i-1}, \gamma \geq 1$. Here F is a fixed cumulative d.f. with $F(0) = 0$. In Kukush (1999) some results concerning asymptotic properties of the MLE in the Nevzorov's record model are obtained, such as consistency, asymptotic normality and efficiency. In that paper the semiparametric and the three-parameter model are studied. The author interprets the observed claims as a stochastically increasing sequence of Fréchet distributed random variables. This idea was first proposed in Pfeifer (1997) with some simulation study.

Here we continue that investigations. The goodness-of-fit test is obtained for the above-mentioned model.

The paper is organized as follows. In Section 2 the main results are presented, and in Section 3 proofs are given.

2. GOODNESS-OF-FIT TEST

Let θ_0 be an interior point of $\Theta \subset \mathbf{R}^d$, and $\hat{\theta}_n$ be a strongly consistent estimator of a parameter θ_0, i.e., $\hat{\theta}_n$ converges to θ_0 a.s., as $n \to \infty$. Consider a random functional $Q_n(\theta) \in C^1(\Theta)$ and suppose that with probability 1 $Q_n(\theta)$ converges uniformly on each compact subset of Θ to a limit functional $Q_\infty(\theta, \theta_0)$, and $Q_\infty(u, v) \in C^1(\Theta \times \Theta)$. Suppose also that $Q_n(\theta) \leq Q_n(\hat{\theta}_n)$ a.s. for all $\theta \in \Theta$ and $Q_\infty(\theta, \theta_0) < Q_\infty(\theta_0, \theta_0)$ for all $\theta \in \Theta,\ \theta \neq \theta_0$.

Theorem 1. *Assume that:*

- *1)* $\begin{pmatrix} \sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0,\theta_0)) \\ \sqrt{n}\ grad\ Q_n(\theta_0) \end{pmatrix} \to \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ *in distribution,* ξ_1 *is a random variable, and* ξ_2 *is a random vector in* $\mathbf{R}^d$*;*

- *2)* $Q_n''(\theta_0) \to S$ *in probability,* S *is a nonsingular matrix;*

- *3)* $\lim_{\epsilon\to 0+} \limsup_n P\{\sup_{\|\theta-\theta_0\|\le\epsilon} \|Q_n''(\theta - Q_n''(\theta_0\| > \delta\} = 0$ *for all* δ*.*

Then $T_n = \sqrt{n}(Q_n(\hat\theta_n) - Q_\infty(\hat\theta_n,\hat\theta_n)) \to \nu = (1, gS^{-1})\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ *in law.* Here $g = \frac{\partial Q_\infty}{\partial v}(\theta_0,\theta_0)$. We regard here and further the derivative vectors as row vectors.

The model in which d.f. F_i has a form $F_i = F(x)^{\gamma^{i-1}}$, and $F(x)$ is unspecified is called *semi-parametric*.

We call *three-parameter model* to be the model in which d.f. F_i has a form $F_i = F(x; A, \alpha)^{\gamma^{i-1}}$, F is Fréchet distribution function with parameters A, α, $F(x) = \exp\left(-(Ax)^{-\alpha}\right),\ i = 1, 2, ...$

Let L be a log-likelihood function, and $Q_n(X,\theta)$ be a normalized log-likelihood function, $Q_n = \frac{1}{n}L$. In the semiparametric model

$$Q_n(\gamma) = \frac{1}{n}\sum_{i=2}^{n}(I_i - p_i^0)\ln\frac{p_i}{1-p_i} + \frac{1}{n}\sum_{i=2}^{n}\Big[p_i^0\ln\frac{p_i}{1-p_i} + \ln(1-p_i)\Big],$$

and in the three-parameter model

$$Q_n(\theta) = \frac{1}{n}L(A,\alpha,\gamma) = \Big(1 - \frac{1}{n}\Big)\frac{\psi}{2} + \ln\alpha - \alpha\ln A + (\alpha+1)\ln A_0 -$$

$$-\frac{\alpha+1}{\alpha_0}\frac{\psi_0}{2}\Big(1-\frac{1}{n}\Big) - \frac{1}{n}\frac{\alpha+1}{\alpha_0}\sum_{i=1}^{n}\ln Z_i - \left(\frac{A_0}{A}\right)^{\alpha}\frac{1}{n}\sum_{i=1}^{n}e^{\frac{i-1}{n}(\psi - \psi_0\frac{\alpha}{\alpha_0})}Z_i^{-\frac{\alpha}{\alpha_0}}.$$

In the semi-parametric model for $\gamma > 1,\ \gamma_0 > 1$ define

$$Q_\infty(\gamma,\gamma_0) = (1-\gamma_0^{-1})\ln(\gamma-1) - \ln\gamma.$$

In the three-parameter model define for $\theta = (A,\alpha,\psi) \in \Theta = (0,\infty)\times(0,\infty)\times\mathbf{R}$, $\psi = n\ln\gamma$, $\theta_0 = (A_0,\alpha_0,\psi_0) \in \Theta$, $\psi_0 = n\ln\gamma_0$.

$$Q_\infty(\theta,\theta_0) = \frac{1}{2}\Big(\psi - \frac{\alpha+1}{\alpha_0}\psi_0\Big) + \ln\alpha A_0 + \alpha\ln\frac{A_0}{A} - \frac{\alpha+1}{\alpha_0}\gamma -$$

$$-\left(\frac{A_0}{A}\right)^{\alpha}\Gamma\Big(1+\frac{\alpha}{\alpha_0}\Big)\frac{e^{\psi - \frac{\alpha}{\alpha_0}\psi_0} - 1}{\psi - \frac{\alpha}{\alpha_0}\psi_0}.$$

Applying the Theorem 1 to semi-parametric and three-parameter models we obtain the following results:

Theorem 2. *In semi-parametric model with* $\gamma_0 > 1$

$$T_n = \sqrt{n}(Q_n(\hat{\gamma}_n) - Q_\infty(\hat{\gamma}_n, \hat{\gamma}_n)) \rightarrow \ N(0, \sigma^2(\gamma_0)) \ \textit{in law. Here}$$

$$\sigma^2(\gamma_0) = \frac{2l_0(l_0\gamma_0^2 - 2l_0\gamma_0 + l_0 + 1)}{\gamma_0^4}, \ l_0 = (\ln(\gamma_0 - 1))^2.$$

Corollary 1. *Let the conditions of Theorem 2 hold. Then*

$$V_n = \frac{T_n}{\sigma(\hat{\gamma}_n)} \rightarrow N(0,1) \textit{ in law.}$$

Theorem 3. *In three-parameter model with* $\gamma_0 > 1$

$$T_n = \sqrt{n}(Q_n(\hat{\theta}_n) - Q_\infty(\hat{\theta}_n, \hat{\theta}_n)) \rightarrow N(0, \sigma^2(\theta_0)) \textit{ in law. Here}$$

$$\sigma^2(\theta_0) = \kappa^T B K B^T \kappa, \ \kappa^T = (1, gS^{-1}), \ g = \left(\frac{1}{A_0}, \quad \frac{1}{2}\frac{\psi_0 + 2\alpha_0 + 2\gamma_e}{\alpha_0^2}, \quad -\frac{1}{2\alpha_0} \right),$$

$$B = \begin{pmatrix} \frac{\alpha_0+1}{\alpha_0} & 1 & 0 & 0 \\ 0 & -\frac{\alpha_0}{A_0} & 0 & 0 \\ 0 & 0 & -\frac{\psi_0}{\alpha_0} & \frac{1}{\alpha_0} \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \frac{\pi^2}{6} & -1 & -\frac{1}{2} & \gamma_e \\ -1 & 1 & \frac{1}{2} & 1-\gamma_e \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1-\gamma_e}{2} \\ \gamma_e & 1-\gamma_e & \frac{1-\gamma_e}{2} & 1 + \frac{\pi^2}{3} + 4\gamma_e + \gamma_e^2 \end{pmatrix},$$

$$S = \begin{pmatrix} -\frac{\alpha_0^2}{A_0^2} & -\frac{1}{2}\frac{\psi_0 - 2 + 2\gamma_e}{A_0} & \frac{1}{2}\frac{\alpha_0}{A_0} \\ -\frac{1}{2}\frac{\psi_0-2+2\gamma_e}{A_0} & -\frac{1}{6}\frac{6\gamma_e^2 - 12\gamma_e - 6\psi_0 + 2\psi_0^2 + \pi^2 + 6 + 6\psi_0\gamma_e}{\alpha_0^2} & \frac{1}{6}\frac{3\gamma_e - 3 + 2\psi_0}{\alpha_0} \\ \frac{1}{2}\frac{\alpha_0}{A_0} & \frac{1}{6}\frac{3\gamma_e-3+2\psi_0}{\alpha_0} & -\frac{1}{3} \end{pmatrix},$$

γ_e is Euler's constant.

Corollary 2. *Let the conditions of Theorem 3 hold. Then*

$$V_n = \frac{T_n}{\sigma(\hat{\theta}_n)} \rightarrow N(0,1) \textit{ in law.}$$

Corollaries 1 and 2 are applied to goodness-of-fit test. In both semi-parametric and three-parameter cases we reject the hypothesis about validity of the model with $\gamma_0 > 1$ if $|V_n| > N_{\alpha/2}$, where $N_{\alpha/2}$ is $\alpha/2$-quantile of normal law, i.e., $P\{N(0,1) > N_{\alpha/2}\} = \alpha/2$.

3. PROOFS

Proof of the Theorem 1.

Before proving the Theorem 1 consider the following three statements:

Lemma 1. *Let $\Theta \subset \mathbf{R^d}$, θ_0 be an interior point of Θ, $\{Q_n(\theta),\ \theta \in \Theta,\ n \geq 1\}$ be sequence of random fields, which are twice differentiable in the neighborhood of θ_0. Let θ_n be a random vector defined by*

$$\theta_n \in \arg\max_{\theta \in \Theta} Q_n(\theta),$$

and suppose that $\theta_n \to \theta_0$ in probability. Assume also that:

- *a) $\sqrt{n}Q_n'(\theta_0)$ converges in law to a random vector γ,*
- *b) $Q_n''(\theta_0) \to S$ in probability, where S is nonsingular nonrandom matrix,*
- *c) For each $\delta > 0$*

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} P\Big(\sup_{\|\theta - \theta_0\| \leq \epsilon} \|Q_n''(\theta) - Q_n''(\theta_0)\| > \delta \Big) = 0.$$

Then $\delta_n := \sqrt{n}(\theta_n - \theta_0) \to -S^{-1}\gamma$ in law.

Proof.

$Q_n'(\theta_n) = 0$ with probability tending to 1, because $\theta_n \in \arg\max_\theta Q_n(\theta)$ and $\theta_n \to \theta_0$ in probability, θ_0 is interior point. Then

$$\frac{\partial Q_n(\theta_n)}{\partial \theta^i} = 0,\ \ \frac{\partial Q_n(\theta_0)}{\partial \theta^i} + \sum_{j=1}^{n} \frac{\partial^2 Q_n(\bar{\theta}_i)}{\partial \theta^i \partial \theta^j}(\theta_n^j - \theta_0^j) = 0,\ \ \bar{\theta}_i \in [\theta_0, \theta_n].$$

Then $\sqrt{n}Q_n'(\theta_0) + Q_n''(\theta_0)\delta_n + R_n = 0$, $R_n = \Lambda_n(\theta_n - \theta_0)\sqrt{n}$, $\Lambda_n = (\Lambda_n^{ij})_{i,j=1}^3$,

$$\Lambda_n^{ij} = \frac{\partial^2 Q_n(\bar{\theta}_i)}{\partial \theta^i \partial \theta^j} - \frac{\partial^2 Q_n(\theta_0)}{\partial \theta^i \partial \theta^j}.$$

So,

$$(Q_n''(\theta_0) + \Lambda_n)\delta_n = -\sqrt{n}Q_n'(\theta_0). \tag{1}$$

Now, $\Lambda_n \to 0$ in probability. Indeed,

$$P\{\|\Lambda_n\| \geq \delta\} \leq P\{\|\theta_n - \theta_0\| > \epsilon\} + P\{\sup_{\|\bar{\theta}_i - \theta_0\| \leq \epsilon} \|\Lambda_n\| \geq \delta\},$$

$$\limsup_{n \to \infty} P\{\|\Lambda_n\| \geq \delta\} \leq \limsup_{n \to \infty} P \sup_{\|\bar{\theta}_i - \theta_0\| \leq \epsilon} \{\|\Lambda_n\| \geq \delta\} \to 0,\ \ \epsilon \to 0+$$

by condition (c). Thus $\Lambda_n \to 0$ in probability. Then in (??) $-\sqrt{n}Q'_n(\theta_0) \to -\gamma$ in law, $Q''_n(\theta_0) + \Lambda_n \to S$ in probability. So, $\delta_n = (Q''_n(\theta_0) + \Lambda_n)^{-1}(-\sqrt{n}Q'_n(\theta_0))$, it holds with probability tending to 1 and $\delta_n \to -S^{-1}\gamma$ in distribution, because S is nonsingular.□

Lemma 2. *Assume that:*

- *1)* $\sqrt{n}Q'_n(\theta_0) \to \xi$ *in distribution,* ξ *is a random vector in* $\mathbf{R}^d$*;*
- *2)* $Q''_n(\theta_0) \to S$ *in probability,* S *is a nonsingular non-random matrix;*
- *3)* $\lim_{\epsilon\to 0+} \limsup_n P\{\sup_{\|\theta-\theta_0\|\le\epsilon} \|Q''_n(\theta - Q''_n(\theta_0\| > \delta\} = 0$ *for all* δ.

Then

$$\sqrt{n}(Q_\infty(\hat{\theta}_n, \hat{\theta}_n) - Q_\infty(\theta_0, \theta_0)) = g\delta_n + o_p(1),\ n \to \infty,\ \textit{where}\ g = \frac{\partial Q_\infty}{\partial v}(\theta_0, \theta_0).$$

Proof. Expand the value $Q_\infty(\hat{\theta}_n, \hat{\theta}_n)$ into Taylor series near the point (θ_0, θ_0).

$$Q_\infty(\hat{\theta}_n, \hat{\theta}_n) = Q_\infty(\theta_0, \theta_0) + \frac{\partial Q_\infty}{\partial u}(\theta_0, \theta_0)(\hat{\theta}_n - \theta_0) + \frac{\partial Q_\infty}{\partial v}(\theta_0, \theta_0)(\hat{\theta}_n - \theta_0) + o(\|\hat{\theta}_n - \theta_0\|).$$

$\theta = \theta_0$ is a maximum point of $Q_\infty(\theta, \theta_0)$. So $\frac{\partial Q_\infty}{\partial u}(\theta_0, \theta_0) = 0$. According to lemma 1, all the conditions of which are satisfied, $\delta_n = \sqrt{n}(\hat{\theta}_n - \theta_0) \to -S^{-1}\xi$. So, $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_p(1)$ and $\sqrt{n}\|\hat{\theta}_n - \theta_0\| = O_p(1)$. Thus $o(\|\hat{\theta}_n - \theta_0\|) = o(\frac{1}{\sqrt{n}})O_p(1)) = o_p(\frac{1}{\sqrt{n}})$. And finally $\sqrt{n}(Q_\infty(\hat{\theta}_n, \hat{\theta}_n) - Q_\infty(\theta_0, \theta_0)) = g\delta_n + o_p(1)$.□

Lemma 3. *Let the conditions of lemma 2 hold.*

Then $\sqrt{n}(Q_n(\hat{\theta}_n) - Q_n(\theta_0)) = o_p(1),\ n \to \infty$.

Proof. Expand the value $Q_n(\theta_0)$ into Taylor series near the point $\theta = \hat{\theta}_n$.

$$Q_n(\theta_0) = Q_n(\hat{\theta}_n) + \frac{dQ_n}{d\theta}(\hat{\theta}_n)(\theta_0 - \hat{\theta}_n) + \frac{d^2Q_n}{d\theta^2}(\bar{\theta})(\theta_0 - \hat{\theta}_n)^2,\ \bar{\theta} \in [\theta_0, \hat{\theta}_n].$$

With probability 1 $\hat{\theta}_n \to \theta_0,\ \theta_0$ is interior point, so $\hat{\theta}_n$ is interior point for $n > n_0(\omega)$. $\theta = \hat{\theta}_n$ is a maximum point of $Q_n(\theta)$. So $\frac{dQ_n}{d\theta}(\hat{\theta}_n) = 0$. Using again Lemma 1 we obtain $\delta_n = \sqrt{n}(\hat{\theta}_n - \theta_0) = O_p(1),\ n \to \infty$. Thus $n(\hat{\theta}_n - \theta_0)^2 = O_p(1),\ n \to \infty$ and $\sqrt{n}(\hat{\theta}_n - \theta_0)^2 = \frac{1}{\sqrt{n}}O_p(1) = o_p(1),\ n \to \infty$. Now, show that $\frac{d^2Q_n}{d\theta^2}(\bar{\theta})$ is stochastically bounded.

$$P\Big\{\Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta})\Big\| \ge C\Big\} \le P\{\|\bar{\theta}_n - \theta_0\| > \epsilon\} + P\Big\{\sup_{\|\bar{\theta}-\theta_0\|\le\epsilon} \Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta})\Big\| \ge C\Big\} \le$$

$$\le P\{\|\hat{\theta}_n - \theta_0\| > \epsilon\} + P\Big\{\sup_{\|\bar{\theta}-\theta_0\|\le\epsilon} \Big(\Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta}) - \frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| + \Big\|\frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\|\Big) \ge C\Big\} \le$$

$$\leq P\{\|\hat{\theta}_n-\theta_0\| > \epsilon\}+P\Big\{\sup_{\|\bar{\theta}-\theta_0\|\leq\epsilon}\Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta})-\frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| \geq \frac{C}{2}\Big\}+P\Big\{\Big\|\frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| \geq \frac{C}{2}\Big\}.$$

Since $\hat{\theta}_n \to \theta_0$, a.s., $n \to \infty$, for any $\epsilon > 0$

$$\limsup_{n\to\infty} P\Big\{\Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta})\Big\| \geq C\Big\} \leq \limsup_{n\to\infty} P\Big\{\sup_{\|\bar{\theta}-\theta_0\|\leq\epsilon}\Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta}) - \frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| \geq \frac{C}{2}\Big\}+$$

$$+\limsup_{n\to\infty} P\Big\{\Big\|\frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| \geq \frac{C}{2}\Big\}.$$

Now, tend ϵ to 0 and use the condition 3) of the lemma.

$$\limsup_{n\to\infty} P\Big\{\Big\|\frac{d^2Q_n}{d\theta^2}(\bar{\theta})\Big\| \geq C\Big\} \leq \limsup_{n\to\infty} P\Big\{\Big\|\frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| \geq \frac{C}{2}\Big\}.$$

According to the condition 2) of the lemma $\Big\|\frac{d^2Q_n}{d\theta^2}(\theta_0)\Big\| = O_p(1)$. By this way we have proved that $\sqrt{n}\frac{d^2Q_n}{d\theta^2}(\bar{\theta})(\theta_0 - \hat{\theta}_n)^2 = o_p(1), \bar{\theta} \in [\theta_0, \hat{\theta}_n]$.

From this the statement of the lemma follows.□

Now, prove Theorem 1. At first mention that condition 1) of Lemma 2 and conditions of Lemma 3 follow from the condition 1 of the Theorem 1.

From Lemma 1 we have: $\delta_n = \sqrt{n}(\hat{\theta}_n - \theta_0) = (Q''_n(\theta_0))^{-1}(-\sqrt{n}Q'_n(\theta_0)) + o_p(1)$, $n \to \infty$; $S\delta_n + \sqrt{n}Q'_n(\theta_0) \to 0$, $n \to \infty$ in probability. Thus

$$\begin{pmatrix} \sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0, \theta_0)) \\ S\delta_n \end{pmatrix} \to \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ in probability.}$$

Apply to this vector the continuous transformation

$$F\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & S^{-1} \end{pmatrix}\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix}.$$

From condition 1) of Lemma 1 using Slutsky lemma we obtain:

$$\begin{pmatrix} \sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0, \theta_0)) \\ \delta_n \end{pmatrix} \to \begin{pmatrix} \xi_1 \\ -S^{-1}\xi_2 \end{pmatrix} \text{ in distribution.}$$

And similarly

$$\begin{pmatrix} \sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0, \theta_0)) \\ g\delta_n \end{pmatrix} \to \begin{pmatrix} \xi_1 \\ -gS^{-1}\xi_2 \end{pmatrix} \text{ in distribution.}$$

In Lemma 3 it was proved that

$$\sqrt{n}(Q_\infty(\hat{\theta}_n, \hat{\theta}_n) - Q_\infty(\theta_0, \theta_0)) = g\delta_n + o_p(1),\ n \to \infty.$$

So,

$$\begin{pmatrix} \sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0, \theta_0)) \\ \sqrt{n}(Q_\infty(\hat{\theta}_n, \hat{\theta}_n) - Q_\infty(\theta_0, \theta_0)) \end{pmatrix} \to \begin{pmatrix} \xi_1 \\ -gS^{-1}\xi_2 \end{pmatrix} \text{ in distribution.}$$

Using Lemma 4 we obtain:

$$\begin{pmatrix} \sqrt{n}(Q_n(\hat{\theta}_n) - Q_\infty(\theta_0, \theta_0)) \\ \sqrt{n}(Q_\infty(\hat{\theta}_n, \hat{\theta}_n) - Q_\infty(\theta_0, \theta_0)) \end{pmatrix} \to \begin{pmatrix} \xi_1 \\ -gS^{-1}\xi_2 \end{pmatrix} \text{ in distribution.}$$

Finally, considering the transformation $G\begin{pmatrix} \upsilon_1 \\ \upsilon_2 \end{pmatrix} = \upsilon_1 - \upsilon_2$ we obtain:

$$\sqrt{n}(Q_n(\hat{\theta}_n) - Q_\infty(\hat{\theta}_n, \hat{\theta}_n)) \to \xi_1 + gS^{-1}\xi_2 = (1, gS^{-1})\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ in distribution.}\Box$$

Proof of the Theorem 2.

We must check three conditions of the theorem 1 and prove that the vector $(1, gS^{-1})\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is normally distributed with parameters $(0, \sigma^2(\theta_0))$.

Check the first condition. Consider $W = \begin{pmatrix} \sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0, \theta_0) \\ \sqrt{n} \text{ grad } Q_n(\theta_0) \end{pmatrix}$. It was proved above that

$$W = \begin{pmatrix} \frac{1}{\sqrt{n}} \sum_{i=2}^n (I_i - p_i^0) \ln \frac{p_i^0}{1-p_i^0} \\ \frac{1}{\sqrt{n}} \sum_{i=2}^n (I_i - p_i^0) \frac{p_i'(\gamma_0)}{p_i^0(1-p_i^0)} \end{pmatrix} = \frac{1}{\sqrt{n}} \sum_{i=2}^n (I_i - p_i^0) \begin{pmatrix} \ln \frac{p_i^0}{1-p_i^0} \\ \frac{p_i'(\gamma_0)}{p_i^0(1-p_i^0)} \end{pmatrix}$$

Denote $A_i(\gamma_0) = \begin{pmatrix} \ln \frac{p_i^0}{1-p_i^0} \\ \frac{p_i'(\gamma_0)}{p_i^0(1-p_i^0)} \end{pmatrix}$. Then Var W $= \frac{1}{n}\sum_{i=1}^n A_i v A_i^T$. Here $v = \text{Var }(I_i - p_i^0) = p_i^0(1 - p_i^0)$. Thus

$$\text{Var } W \to \begin{pmatrix} (\ln(\gamma_0 - 1))^2 \frac{(\gamma_0 - 1)}{\gamma_0^2} & -\frac{\ln(\gamma_0 - 1)}{\gamma_0^2} \\ -\frac{\ln(\gamma_0 - 1)}{\gamma_0^2} & \frac{1}{\gamma_0^2(\gamma_0 - 1)} \end{pmatrix} := K.$$

Now, the second condition of the theorem 1 holds, as $Q_n''(\gamma_0) \to S = -\frac{1}{\gamma_0^2(\gamma_0-1)}$, $n \to \infty$ in probability.

It is easy to see that $g = \frac{\partial Q_\infty}{\partial v}(\theta_0, \theta_0) = \frac{\ln \gamma_0 - 1}{\gamma_0^2}$.

The third condition of the theorem holds as well. Therefore, applying the theorem 1 we obtain:

$$T_n = \sqrt{n}(Q_n(\hat{\gamma}_n) - Q_\infty(\hat{\gamma}_n, \hat{\gamma}_n)) \to \; N(0, \sigma^2(\gamma_0)) \text{ in law.}$$

$$\sigma^2(\gamma_0) = \kappa K \kappa^T, \; \kappa = (1, gS^{-1})^T.$$

$$\sigma^2(\gamma_0) = \frac{2l_0(l_0\gamma_0^2 - 2l_0\gamma_0 + l_0 + 1)}{\gamma_0^4},\ l_0 = (\ln(\gamma_0 - 1))^2.\Box$$

Proof of the Theorem 3.

(i) Reparametrization. Consider the maximum likelihood function for the observed data set $X_1, ..., X_n$, $L(A, \alpha, \gamma) = \sum_{i=1}^n \ln f(x_i; A, \alpha, \gamma)$. Here $f_i(x; A, \alpha, \gamma)$ is density function of described above distribution, $f_i(x; A, \alpha, \gamma) = \frac{\partial F_i(x;A,\alpha,\gamma)}{\partial x}$. Let $\psi = n \ln \gamma$. Rewrite this functional using new parameter $\theta = (A, \alpha, \psi)$ (the same is for true values), $\theta \in (0, \infty) \times (0, \infty) \times \mathbf{R}$.

$$Q_n(\theta) = \frac{1}{n} L(A, \alpha, \gamma) = (1 - \frac{1}{n})\frac{\psi}{2} + \ln(\alpha A^{-\alpha}) - \frac{1}{n}(\alpha+1)\sum_{i=1}^n \ln X_i - \frac{1}{n}\sum_{i=1}^n e^{\frac{i-1}{n}\psi}(AX_i)^{-\alpha}$$

Rewrite the functional $Q_n(\theta)$ using the i.i.d. sequence $Z_i = (A_0 X_i)^{\alpha_0}(\gamma_0^{-\frac{1}{\alpha_0}})^{i-1},\ i = 1, 2, ...,$

$$Q_n(\theta) = \frac{1}{n} L(A, \alpha, \gamma) = \Big(1 - \frac{1}{n}\Big)\frac{\psi}{2} + \ln\alpha - \alpha \ln A + (\alpha + 1)\ln A_0 -$$
$$-\frac{\alpha+1}{\alpha_0}\frac{\psi_0}{2}\Big(1 - \frac{1}{n}\Big) - \frac{1}{n}\frac{\alpha+1}{\alpha_0}\sum_{i=1}^n \ln Z_i - \left(\frac{A_0}{A}\right)^\alpha \frac{1}{n}\sum_{i=1}^n e^{\frac{i-1}{n}(\psi - \psi_0\frac{\alpha}{\alpha_0})} Z_i^{-\frac{\alpha}{\alpha_0}}.$$

We obtain

$$Q_n(\theta) = \frac{n-1}{2n}\Big(\psi - \frac{\alpha+1}{\alpha_0}\psi_0\Big) + \ln\alpha A_0 + \alpha(\ln\frac{A_0}{A}) -$$
$$-\frac{\alpha+1}{\alpha_0}\frac{1}{n}\sum_{i=1}^n \ln Z_i - \left(\frac{A_0}{A}\right)^\alpha \frac{1}{n}\sum_{i=1}^n e^{\frac{i-1}{n}(\psi - \psi_0\frac{\alpha}{\alpha_0})} Z_i^{-\frac{\alpha}{\alpha_0}} \qquad (2)$$

(ii) Limit functional. Consider θ belong to a compact subset of Θ. Uniformly on this set we have

$$Q_n(\theta) = Q_\infty(\theta) + R_1 + R_2 + o(1),$$

with the limit functional

$$Q_\infty(\theta, \theta_0) = \frac{1}{2}\Big(\psi - \frac{\alpha+1}{\alpha_0}\psi_0\Big) + \ln\alpha A_0 + \alpha\ln\frac{A_0}{A} - \frac{\alpha+1}{\alpha_0}\gamma -$$
$$-\left(\frac{A_0}{A}\right)^\alpha \Gamma\Big(1 + \frac{\alpha}{\alpha_0}\Big)\frac{e^{\psi - \frac{\alpha}{\alpha_0}\psi_0} - 1}{\psi - \frac{\alpha}{\alpha_0}\psi_0} \qquad (3)$$

and

$$R_1(\theta) = -\frac{1}{n}\frac{\alpha+1}{\alpha_0}\sum_{i=1}^n\Big(\ln Z_i - E\ln Z_i\Big); \qquad (4)$$

$$R_2(\theta) = -\frac{1}{n}\left(\frac{A_0}{A}\right)^{\alpha}\sum_{i=1}^{n} e^{\frac{i-1}{n}(\psi-\psi_0\frac{\alpha}{\alpha_0})}\Big(Z_i^{-\frac{\alpha}{\alpha_0}} - EZ_i^{-\frac{\alpha}{\alpha_0}}\Big). \tag{5}$$

With probability 1 $Q_n(\theta)$ converges to $Q_\infty(\theta,\theta_0)$ uniformly.

(iii) We must check the conditions of Theorem 1 and prove that $(1, gS^{-1})\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ is normally distributed with mean 0 and variance $\sigma^2(\theta_0)$.

Check the first condition of the Theorem 1:

Consider $W = \begin{pmatrix}\sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0,\theta_0)\\ \sqrt{n}\ \mathrm{grad}\ Q_n(\theta_0)\end{pmatrix}$. It is easy to see that $\sqrt{n}(Q_n(\theta_0) - Q_\infty(\theta_0,\theta_0) = \sqrt{n}(R_1(\theta_0) + R_2(\theta_0))$. Here

$$R_1(\theta_0) = -\frac{\alpha_0+1}{\alpha_0}\frac{1}{n}\sum_{i=1}^{n}(\ln z_i - E\ln z_i);$$

$$R_2(\theta_0) = R_2 = -\frac{1}{n}\sum_{i=1}^{n} z_i^{-1} - Ez_i^{-1}.$$

Another part of W has the form:

$$\sqrt{n}Q_n'(\theta_0) =$$

$$= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix} -\frac{\alpha_0}{A_0}(z_i^{-1} - Ez_i^{-1})\\ \frac{1}{\alpha_0}[((1-z_i^{-1})\ln z_i - \psi_0\frac{i-1}{n}z_i^{-1}) - E((1-z_i^{-1})\ln z_i - \psi_0\frac{i-1}{n}z_i^{-1})]\\ \frac{i-1}{n}(z_i^{-1} - Ez_i^{-1})\end{pmatrix}$$

Thus

$$W = -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{pmatrix} (\frac{\alpha_0+1}{\alpha_0}\ln z_i + z_i^{-1}) - E(\frac{\alpha_0+1}{\alpha_0}\ln z_i + z_i^{-1})\\ -\frac{\alpha_0}{A_0}(z_i^{-1} - Ez_i^{-1})\\ \frac{1}{\alpha_0}[((1-z_i^{-1})\ln z_i - \psi_0\frac{i-1}{n}z_i^{-1}) - E((1-z_i^{-1})\ln z_i - \psi_0\frac{i-1}{n}z_i^{-1})]\\ \frac{i-1}{n}(z_i^{-1} - Ez_i^{-1})\end{pmatrix}$$

Consider the vector

$$\zeta_i = \begin{pmatrix}\ln z_i - E\ln z_i\\ z_i^{-1} - Ez_i^{-1}\\ (1-z_i^{-1})\ln z_i - E((1-z_i^{-1})\ln z_i)\end{pmatrix}$$

with

$$\Gamma = \mathrm{Var}\ \zeta_i = \begin{pmatrix} a & b & c\\ b & d & e\\ c & e & f\end{pmatrix}.$$

Consider also continuous linear transformation

$$A_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{i-1}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \zeta_i = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} \ln z_i - E \ln z_i \\ z_i^{-1} - E z_i^{-1} \\ \frac{i-1}{n}(z_i^{-1} - E z_i^{-1}) \\ (1 - z_i^{-1}) \ln z_i - E((1 - z_i^{-1}) \ln z_i) \end{pmatrix}.$$

The corresponding covariance matrix has the form

$$\text{Var}\,\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \zeta_i\Big) = \frac{1}{n} \sum_{i=1}^{n} A_i \Gamma A_i^T = \frac{1}{n} \sum_{i=1}^{n} \begin{pmatrix} a & b & \frac{i-1}{n} b & c \\ b & d & \frac{i-1}{n} d & e \\ \frac{i-1}{n} b & \frac{i-1}{n} d & \left(\frac{i-1}{n}\right)^2 d & \frac{i-1}{n} e \\ c & e & \frac{i-1}{n} e & f \end{pmatrix} =$$

$$= \begin{pmatrix} a & b & b\frac{S_2}{n^2} & c \\ b & d & db\frac{S_2}{n^2} & e \\ b\frac{S_2}{n^2} & d\frac{S_2}{n^2} & d\frac{S_3}{n^3} & e\frac{S_2}{n^2} \\ c & e & e\frac{S_2}{n^2} & f \end{pmatrix} \longmapsto \begin{pmatrix} a & b & \frac{b}{2} & c \\ b & d & \frac{d}{2} & e \\ \frac{b}{2} & \frac{d}{2} & \frac{d}{3} & \frac{e}{2} \\ c & e & \frac{e}{2} & f \end{pmatrix} := K.$$

Here $S_k = \sum_{i=1}^{n} (i-1)^{k-1},\ k > 1,\ \frac{S_2}{n^2} = \frac{n(n-1)}{2n^2} \to \frac{1}{2};\ \ \frac{S_3}{n^3} = \frac{n(n-1)(2n-1)}{6} \to \frac{1}{3}.$

Now by multivariate CLT $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \zeta_i \to \rho$, ρ is normal distributed with mean 0 and covariance matrix K. Indeed, check Lyapunov condition for this vector. It is sufficient to check it for the components of the vector. Check it only for third component of the vector $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \zeta_i$. (Checking the condition for the other components is obvious.) Let

$$\xi_i = z_i^{-1} - E z_i^{-1},\ \xi_{ni} = \frac{i-1}{n\sqrt{n}} \xi_i,\ S_n = \sum_{i=1}^{n} \xi_{ni} = \Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_i \zeta_i\Big)_3.$$

Now,

$$s_n^2 = \text{Var}\ S_n = \frac{1}{n} \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \to \frac{1}{3}.$$

Check Lyapunov condition:

$$\frac{1}{s_n^3} \sum_{i=1}^{n} E|\xi_{nk}|^3 \to 0,\ n \to \infty.$$

Indeed,

$$\frac{1}{(n\sqrt{n})^3}\sum_{i=1}^{n} E|\xi_i|^3 = \frac{S_4(n)}{n^4\sqrt{n}}E|\xi_1|^3 \to 0,\ n\to\infty,\ S_4(n)=\sum_{i=1}^{n}(i-1)^3.$$

Consider another transformation:

$$B = \begin{pmatrix} \frac{\alpha_0+1}{\alpha_0} & 1 & 0 & 0 \\ 0 & -\frac{\alpha_0}{A_0} & 0 & 0 \\ 0 & 0 & -\frac{\psi_0}{\alpha_0} & \frac{1}{\alpha_0} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

It is easy to see that $W = -B\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} A_i\zeta_i\Big)$. And we obtain that $W \to N(0,T)$, where $T = \mathrm{Var}\ W = BKB^T$. Obviously, $(1, gS^{-1})\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}$ is normally distributed with mean 0 and variance $\sigma^2(\theta_0)$, where $\sigma^2(\theta_0) = \kappa^T BKB^T\kappa$, $\kappa^T = (1, gS^{-1})$.

Now, check the second condition of the theorem 1.

Consider $Q_n(\theta) = Q_\infty(\theta,\theta_0) + R_1(\theta) + R_2(\theta)$. Direct calculations show that

$$(Q_\infty)''(\theta_0,\theta_0) = \begin{pmatrix} -\frac{\alpha_0^2}{A_0^2} & -\frac{1}{2}\frac{\psi_0-2+2\gamma_e}{A_0} & \frac{1}{2}\frac{\alpha_0}{A_0} \\ -\frac{1}{2}\frac{\psi_0-2+2\gamma_e}{A_0} & -\frac{1}{6}\frac{6\gamma_e^2-12\gamma_e-6\psi_0+2\psi_0^2+\pi^2+6+6\psi_0\gamma_e}{\alpha_0^2} & \frac{1}{6}\frac{3\gamma_e-3+2\psi_0}{\alpha_0} \\ \frac{1}{2}\frac{\alpha_0}{A_0} & \frac{1}{6}\frac{3\gamma_e-3+2\psi_0}{\alpha_0} & -\frac{1}{3} \end{pmatrix} := S.$$

It is easy to see that $R_1''(\theta) \equiv 0$ for all $\theta \in \Theta$.

Now consider $R_2''(\theta_0)$. Every element of this matrix consists of linear combination of the following expressions:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1} - Ez_i^{-1}),\ \frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1}\ln z_i - E(z_i^{-1}\ln z_i)),$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1}\ln^2 z_i - E(z_i^{-1}\ln^2 z_i)),\ k=0,1,2.$$

Thus $R_2''(\theta_0) \to 0,\ n\to\infty$, in probability. Consider, for example, the most interesting term

$$\frac{\partial^2 R_2}{\partial\psi^2}(\theta) = -\left(\frac{A_0}{A}\right)^{\alpha}\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^2}{n^2}e^{\frac{i-1}{n}(\psi-\psi_0\frac{\alpha}{\alpha_0})}\Big({z_i}^{\frac{\alpha}{\alpha_0}} - E{z_i}^{\frac{\alpha}{\alpha_0}}\Big),$$

$$\frac{\partial^2 R_2}{\partial\psi^2}(\theta_0) = \frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^2}{n^2}\Big({z_i}^{-1} - E{z_i}^{-1}\Big).$$

But

$$\operatorname{Var}\frac{\partial^2 R_2}{\partial\psi^2}(\theta_0)=\frac{1}{n^2}\frac{1-3n+2n^2}{n}\mathrm{Const}\asymp\frac{1}{n}\to 0,\ n\to\infty.$$

And $\frac{\partial^2 R_2}{\partial\psi^2}(\theta_0)\to 0,\ n\to\infty$ in probability.

Finally, check the last condition of the theorem 1. Consider $Q_n'''(\theta_0)$. Every element of this tensor consists of linear combination of the following expressions:

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1}-Ez_i^{-1}),\ \frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1}\ln z_i-E(z_i^{-1}\ln z_i)),$$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1}\ln^2 z_i-E(z_i^{-1}\ln^2 z_i)),\ \sum_{i=1}^{n}\frac{(i-1)^k}{n^k}(z_i^{-1}\ln^3 z_i-E(z_i^{-1}\ln^3 z_i)),$$

$k=0,1,2,3$. Consider the most interesting: $\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^3}{n^3}\Big({z_i}^{-1}-E{z_i}^{-1}\Big)$.

$$\operatorname{Var}\left\{\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)^3}{n^3}\Big({z_i}^{-1}-E{z_i}^{-1}\Big)\right\}=\frac{1}{n^2}\frac{1-2n+n^2}{n}\mathrm{Const}\asymp\frac{1}{n}\to 0,\ n\to\infty.$$

Therefore there exists a constant $C=C(\omega)$ s.t. $|Q_n'''(\theta_0)|<C$. From this statement the third condition follows.$\square$

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