Statistical Inference with Fractional Brownian Motion

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Abstract. We give a test between two complex hypothesis; namely we test whether a fractional Brownian motion (fBm) has a linear trend against a certain non-linear trend. We study some related questions, like goodness-of-fit test and volatility estimation in these models.

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1. Introduction

1.1. SETUP

A fractional Brownian motion (fBm) $Z = Z^H$ is a self-similar Gaussian process with index $H \in (0, 1)$: it is a continuous Gaussian process with stationary increments, defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, with the properties

- $Z_0 = 0$.
- $\mathbb{E} Z_t = 0$ for every $t \geqslant 0$.
- $\mathbb{E} Z_t Z_s = 1/2(t^{2H} + s^{2H} - |s - t|^{2H})$ for every $s, t \geqslant 0$.

FBm is not a semimartingale, if $H \neq 1/2$. There are several approaches to define stochastic integrals w.r.t. fBm and they are briefly mentioned in connection to the linear Equation (1.1) in Section 1.2.

How the stochastic integral is defined has an impact on modeling with fBm. Namely, the so-called Riemann–Stieltjes integral contributes to the mean rate of signal, but the so-called Skorohod integral does not contribute to the mean rate (see Duncan et al., 2000, p. 583). Concerning the use of fBm as a model in finance the meaning of the two different integrals is still under discussion (for more details see Hu and Øksendal, 2003; Sottinen and Valkeila, 2003).

In the context of the linear Equation (1.1) this means that the difference is a certain non-linear drift. This test is developed in Section 4, where the observations are discrete and the intensity σ of the fBm is unknown. In Sections 2 and 3 we give the necessary background for this test. In addition we give some results on goodness-of-fit in Sections 5 and 6 and on volatility estimation in Section 7. The simulation study of the proposed tests is a separate and non-trivial problem and will be not discussed here. Auxiliary computations are gathered in Appendix.

We suppose throughout the paper that index H is known.

1.2. LINEAR SDE W.R.T. FBM

Consider the following linear SDE

$$\mathrm{d}Y_t = \mu Y_t\,\mathrm{d}t + \sigma Y_t\,\mathrm{d}Z_t, \qquad Y_0 = y_0, \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

where μ and $\sigma > 0$ are constants.

Assume that the Hurst index H satisfies $H \in (1/2, 1)$.

The linear stochastic differential equation (1.1) can be written in integral form as

$$Y_t = y_0 + \mu \int_0^t Y_s\,\mathrm{d}s + \sigma \int_0^t Y_s\,\mathrm{d}Z_s. \tag{1.2}$$

There are at least two possibilities to understand the stochastic integral in the last term of the right hand side of (1.2): as a pathwise integral or as a Skorohod integral. The first uses regularity properties of fBm: although the process Z is not of bounded variation on compacts, one can use the property that it has zero quadratic variation, or the property that it has zero p-variation for $p > H$ as in (Dudley and Norvaisa, 1999), or the property that it is Hölder continuous with $\beta < H$ on compacts as in (Zähle, 1998) to define the stochastic integral as a Riemann–Stieltjes integral. The second is based on Malliavin calculus (Decreusefond and Üstünel, 1999) or on the 'fractional' white noise calculus with respect to fBm (Duncan et al., 2000). According to the first approach the linear Equation (1.1) has a unique solution (in some smooth class of functions)

$$Y_t = y_0 \exp(\mu t + \sigma Z_t). \tag{1.3}$$

According to the second approach the linear Equation (1.1) has a solution

$$Y_t = y_0 \exp\left(\mu t + \sigma Z_t - \frac{1}{2}\sigma^2 t^{2H}\right); \tag{1.4}$$

the reason for the difference between (1.3) and (1.4) is that the Skorohod integral have zero expectation. But we will not go into more details here. We want to note that the only difference of the solutions of the Equations (1.3) and (1.4) is a non-linear trend in (1.4).

1.3. NOTATION AND DEFINITIONS

Denote by Γ the gamma function

$$\Gamma(p) = \int_0^\infty x^{p-1}\,\mathrm{e}^{-x}\,\mathrm{d}x, \quad p > 0$$

and put $B(\mu, \nu) = \Gamma(\mu)\Gamma(\nu)/\Gamma(\mu + \nu)$.

If X is a process and $I\!F = (F_t)_{t \geqslant 0}$ a filtration, the notation $X \in I\!F$ means that X_t is F_t measurable for each $t \geqslant 0$.

If Y is a square integrable martingale, then the angle bracket process of Y is denoted by $\langle Y \rangle$.

For $t > 0$ put $\pi^n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ and $|\pi^n| = \max\{t_i - t_{i-1} : t_i \in \pi^n\}$. To simplify notation, we write π instead of π^n.

The notation $X_n = o_{I\!P}(1)$ means that $X_n \xrightarrow{I\!P} 0$, $X_n = O_{I\!P}(1)$ means that $\lim_{C\to\infty} \limsup_n I\!P\{|X_n| \geqslant C\} = 0$.

Let $(\Omega, F, I\!P)$ be a complete probability space equipped with the filtration $I\!F = (F_t)_{t \geqslant 0}$. Denote by $(W_t, F_t; t \geqslant 0)$ a standard Brownian motion and by $(Z_t^H, F_t; t \geqslant 0)$ the fBm with self-similarity index $H \in (0, 1)$. We write Z instead of Z^H below. We shall use the following constants

$$B_1 \doteq B\big(\tfrac{3}{2} - H, \tfrac{3}{2} - H\big), \qquad B_2 \doteq B\big(H + \tfrac{1}{2}, \tfrac{3}{2} - H\big),$$
$$B_3 = B\big(H - \tfrac{1}{2}, \tfrac{3}{2} - H\big), \qquad C_0 \doteq \tfrac{1}{2}\big(\big(H - \tfrac{1}{2}\big)H(1 - H)B_1 B_3\big)^{-1/2}.$$

Consider the kernel $z(t, s) \doteq C_0 s^{1/2-H}(t - s)^{1/2-H}$, when $s \in (0, t)$, and $z(t, s) = 0$ otherwise.

We have the following result from (Norros et al., 1999):

$$\int_0^t z(t, s)\,\mathrm{d}Z_s = \int_0^t s^{1/2-H}\,\mathrm{d}W_s, \tag{1.5}$$

where the left integral exists as a pathwise integral w.r.t. fBm Z (Norros et al., 1999, p. 574).

2. The Density Process for fBm with Different Drifts

2.1. TESTING PROBLEM

Assume that $H \in (1/2, 1)$. For a fixed $\mu \in I\!R$ let $I\!P_{\mu,\sigma,\sigma}$ be the distribution of the (observed) process

$$X_t \doteq \sigma Z_t + \mu t - \tfrac{1}{2}\sigma^2 t^{2H}, \quad 0 \leqslant t \leqslant T \tag{2.1}$$

in the space of $C_{[0,T]}$ of continuous functions. Similarly, $I\!P_{\mu,\sigma}$ is the distribution of the process

$$X_t \doteq \sigma Z_t + \mu t, \quad 0 \leqslant t \leqslant T \tag{2.2}$$

in the space $C_{[0,T]}$.

Suppose now that we observe a trajectory of the process X_t, $0 \leqslant t \leqslant T$ in the space $C_{[0,T]}$. Denote by $I\!P_X$ the law of X. We want to test the following complex hypothesis

H $P_X \in \{I\!P_{\mu,\sigma}\colon \mu \in I\!R, \sigma \in I\!R_+\}$

against the complex alternative

$$\mathbf{A} \quad P_X \in \{\mathbb{P}_{\mu,\sigma,\sigma}\colon \mu \in \mathbb{R}, \sigma \in \mathbb{R}_+\}, \qquad \mathbb{R}_+ \doteq (0, \infty).$$

We give an interpretation of the above two hypothesis: if we accept **H**, this means that noise affects the signal, but if we accept **A**, then noise does not affect the signal.

2.2. DENSITY PROCESSES

From the point of view of the general theory, models of observation (2.1), (2.2) are equivalent to the classical models

$$\widetilde{X}_t = \sigma M_t + \mu C_0 B_1 t^{2-2H} - \sigma H B_2 t \tag{2.3}$$

and

$$\widetilde{X}_t = \sigma M_t + \mu C_0 B_1 t^{2-2H}, \tag{2.4}$$

where $\widetilde{X}_t = \int_0^t z(t,s)\,\mathrm{d}X_s$ (see also Le Breton, 1998; Kleptsyna et al., 2000).

Introduce the following density processes (Radon–Nikodym derivatives) based on the observed trajectory X:

$$f_1(X\colon \mu,\sigma,\sigma) \doteq \frac{\mathrm{d}\mathbb{P}_{\mu,\sigma,\sigma}}{\mathrm{d}\mathbb{P}_{0,\sigma}}(X) \tag{2.5}$$

and

$$f_2(X\colon \mu,\sigma) \doteq \frac{\mathrm{d}\mathbb{P}_{\mu,\sigma}}{\mathrm{d}\mathbb{P}_{0,\sigma}}(X). \tag{2.6}$$

THEOREM 2.1. *Assume that we observe X on the interval* $[0, T]$. *We have*

$$f_1(X\colon \mu,\sigma,\sigma) = \exp\left[\alpha\frac{\mu}{\sigma^2}\widetilde{X}_T - \frac{\beta}{\sigma}\widetilde{X}_T^1 - \gamma\frac{\mu^2}{\sigma^2}T^{2-2H} + \delta\mu T - \eta\sigma^2 T^{2H}\right] \tag{2.7}$$

and

$$f_2(X\colon \mu,\sigma) = \exp\left[\alpha\frac{\mu}{\sigma^2}\widetilde{X}_T - \gamma\frac{\mu^2}{\sigma^2}T^{2-2H}\right], \tag{2.8}$$

where

$$\widetilde{X}_t^1 = \int_0^t s^{2H-1}\,\mathrm{d}\widetilde{X}_s, \qquad \alpha = 2C_0B_1(1-H), \qquad \beta = C_0HB_2,$$
$$\gamma = (1-H)C_0^2B_1^2, \qquad \delta = 2(1-H)C_0^2HB_1B_2, \qquad \eta = \tfrac{1}{4}C_0^2HB_2^2. \tag{2.9}$$

Proof. Follows immediately from (2.1) to (2.6) and classical Girsanov theorem. □

3. Observations Based on the Whole Trajectory with σ Known

In this section we show, how to test hypothesis **A** against the alternative **H**, when σ is known and the whole trajectory $\{X_t\colon t \in [0, T]\}$ is observed. We can use the likelihood ratio to test this [for the likelihood ratio see (Borovkov, 1984, p. 349)].

In this problem the likelihood ratio $l(X.) = l(X.|\sigma)$ has a form

$$l(X.|\sigma) \doteq \frac{\sup_{\mu\in I\!R} f_1(X; \mu, \sigma, \sigma)}{\sup_{\mu\in I\!R} f_2(X; \mu, \sigma)}. \tag{3.1}$$

Note that in (3.1) both upper bounds are attained, since the densities f_1 and f_2 are quadratic functions of μ.

More precisely, we have

$$\sup_{\mu\in I\!R} f_1(X; \mu, \sigma, \sigma) = \exp\left\{\frac{1-H}{\sigma^2 T^{2-2H}}(\widetilde{X}_t^1)^2 - \alpha_1 T^{2H-1}\widetilde{X}_t + {} \right. \\ \left. {} + \alpha_1 \int_0^T s^{2H-2}\widetilde{X}_s\, \mathrm{d}s - \beta_1\sigma^2 T^{2H}\right\}$$

and the value of μ giving maximal value in (3.2) is

$$\widehat{\mu}_A \doteq \frac{1}{C_0 B_1 T^{2-2H}}(\widetilde{X}_T + HB_2C_0T\sigma^2); \tag{3.2}$$

and for the denominator in (3.1)

$$\sup_{\mu\in I\!R} f_2(X; \mu, \sigma) = \exp\left(\frac{1-H}{\sigma^2 T^{2-2H}}\widetilde{X}_T^2\right)$$

an the maximum in (3.3) is achieved by

$$\widehat{\mu}_H \doteq \frac{1}{C_0 B_1 T^{2-2H}}\widetilde{X}_T, \tag{3.3}$$

where

$$\alpha_1 \doteq (2H-1)HC_0B_2, \qquad \beta_1 \doteq H\big(H - \tfrac{1}{2}\big)^2 C_0^2 B_2^2.$$

We obtain the following theorem as a direct consequence of (3.2) and (3.3):

THEOREM 3.1. *The likelihood* $l(X.|\sigma)$ *from* (3.1) *has representation*

$$l(X.|\sigma) = \exp\left(-\frac{\alpha_1}{T^{1-2H}}\widetilde{X}_T + \alpha_1\int_0^T s^{2H-2}\widetilde{X}_s\,\mathrm{d}s - \beta_1\sigma^2T^{2H}\right).$$

Remark 3.1. Note that in the case when $H = 1/2$ we have $l(X.|\sigma) = 1$. It means that our method does not work in this case, because the drift $(-\sigma^2 t/2)$ has the same order in t that μt, and we can not distinguish them. Therefore our method works worse if H is close to $1/2$.

Next we describe the testing procedure. Given a confidence level $1 - \alpha$, $\alpha \in (0, 1/2)$, consider the critical areas defined by $K_1 \doteq \{X_\cdot : l(X_\cdot|\sigma) \geqslant K_\alpha\}$ and $K_2 \doteq \{X_\cdot : l(X_\cdot|\sigma) < k_\alpha\}$. The critical values $0 < k_\alpha \leqslant K_\alpha$ are chosen in such a way that we have

$$\sup_{\mu \in I\!R} I\!P_{\mu,\sigma}(K_1) \leqslant \alpha, \qquad \sup_{\mu \in I\!R} I\!P_{\mu,\sigma,\sigma}(K_2) \leqslant \alpha. \tag{3.4}$$

The test is now clear: if $X_\cdot \in K_1$ we accept **A**, if $X_\cdot \in K_2$ we accept **H**. If $l(X_\cdot|\sigma) \in [k_\alpha, K_\alpha)$ then no hypothesis is accepted. The inequalities (3.4) show that the probabilities of so called first and second kind of errors will not exceed the level α.

Next we compute the critical values K_α, k_α. To compute K_α recall that under **H** the process X has the same distribution as the process $\sigma Z_t + \mu t$. Similarly, to compute k_α we use the fact that under **A** the process X has the same distribution as the process $\sigma Z_t + \mu t - (\sigma^2/2)t^{2H}$.

We have that

$$l(\sigma Z_\cdot + \mu \cdot) = \exp\left(-\alpha_1 \sigma T^{2H-1} M_T + \alpha_1 \sigma \int_0^T s^{2H-2} M_s \,\mathrm{d}s - \beta_1 \sigma^2 T^{2H} \right)$$

and

$$l\left(\sigma Z_\cdot + \mu \cdot - \frac{\sigma^2}{2} \cdot \right) = \exp\left(-\alpha_1 \sigma T^{2H-1} M_T + \alpha_1 \sigma \int_0^T s^{2H-2} M_s \,\mathrm{d}s + \beta_1 \sigma^2 T^{2H} \right).$$

Hence, we have that

$$I\!P_{\mu,\sigma}(K_1) = I\!P\left(-\alpha_1 \sigma T^{2H-1} M_T + \alpha_1 \sigma \int_0^T s^{2H-2} M_s \,\mathrm{d}s \geqslant \log K_\alpha + \beta_1 \sigma^2 T^{2H} \right). \tag{3.5}$$

The random variable in the above expression is Gaussian with mean zero and variance

$$v^2 = 2\beta_1 \sigma^2 T^{2H}.$$

Therefore, by (3.4)

$$I\!P_{\mu,\sigma}(K_1) = 1 - \Phi\left(\frac{\log K_\alpha}{v} + \frac{v}{2} \right), \tag{3.6}$$

where Φ is the distribution function of standard normal distribution. If ξ_α is such that $1 - \Phi(\xi_\alpha) = \alpha$, then $K_\alpha \geqslant \exp\{v\xi_\alpha - v^2/2\}$.

Similarly,

$$I\!P_{\mu,\sigma,\sigma}(K_2) = 1 - \Phi\left(\frac{1}{v}\log\left(\frac{1}{k_\alpha}\right) + \frac{v}{2}\right); \tag{3.7}$$

that is, $k_\alpha \leqslant \exp\{-v\xi_\alpha + v^2/2\}$. Finally, we can choose $K_\alpha = \max(1, \exp\{v\xi_\alpha - v^2/2\})$, $k_\alpha = K_\alpha{}^{-1}$.

4. Discretely Observed Trajectory, and σ Unknown

Assume now that we observe the process X discretely and the intensity σ of the fractal noise is unknown. We replace the parameter σ in $l(X.|\sigma)$ with a consistent estimate $\widehat{\sigma}_n$, where n is the number of time points, and instead of the stochastic integrals w.r.t. X we will use sums in terms of the increments of X. We obtain a quasi-likelihood ratio, which is constructed from the observations. The critical values will be computed uniformly w.r.t. all possible values of μ and σ. We will give an asymptotic description of the critical levels.

First, choose the critical values independently of the parameter σ. For $K_\alpha \geqslant 1$ we have that

$$\frac{1}{v}\log K_\alpha + \frac{v}{2} \geqslant 2\sqrt{\frac{1}{2}\log K_\alpha} = \sqrt{2\log K_\alpha},$$

and from (3.6)

$$I\!P_{\mu,\sigma}(K_1) \leqslant 1 - \Phi(\sqrt{2\log K_\alpha}).$$

Take $K_\alpha^* \doteq e^{\xi_\alpha^2/2}$ and put $K_1^* \doteq \{X. : l(X.) \geqslant K_\alpha^*\}$. Then we have

$$\sup_{\mu,\sigma>0} I\!P_{\mu,\sigma}(K_1^*) \leqslant \alpha. \tag{4.1}$$

Similarly, using (3.7) and taking $k_\alpha^* = e^{-\xi_\alpha^2/2}$ and if $K_2^* \doteq \{X. : l(X.) \leqslant k_\alpha^*\}$ we will have

$$\sup_{\mu,\sigma>0} I\!P_{\mu,\sigma,\sigma}(K_2^*) \leqslant \alpha. \tag{4.2}$$

Put

$$K_0^* \doteq \{X.: k_\alpha^* < l(X.) < K_\alpha^*\};$$

note that K_0^* is (a conservative variant of) the region, where neither the hypothesis **H** nor the hypothesis **A** is accepted. Let $C_3 \doteq \sqrt{H/2}\,(2H-1)C_0B_2\sigma$.

THEOREM 4.1. *Assume that $T > (\sqrt{2}\xi_\alpha/C_3)^{1/H}$. Then we have that*

$$\sup_{\mu,\sigma} I\!P_{\mu,\sigma}(K_0^*) \leqslant \frac{4}{C_3}T^{-H}\exp\left\{-\frac{C_3^2T^{2H}}{32}\right\} \tag{4.3}$$

and

$$\sup_{\mu,\sigma,\sigma} I\!P_{\mu,\sigma,\sigma}(K_0^*) \leqslant \frac{4}{C_3} T^{-H} \exp\left\{ -\frac{C_3^2 T^{2H}}{32} \right\}. \tag{4.4}$$

Proof. We have that

$$I\!P_{\mu,\sigma}(K_0^*) \leqslant I\!P_{\mu,\sigma}(\{X_\cdot : l(X_\cdot) > k_\alpha^*\}) = 1 - \Phi\left(\frac{v}{2} + \frac{1}{v} \log k_\alpha^* \right). \tag{4.5}$$

We have the following inequality for $x > 0$

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{e}^{-u^2/2}\, \mathrm{d}u \leqslant \frac{1}{\sqrt{2\pi}\, x} \mathrm{e}^{-x^2/2}.$$

Apply this to (4.5) with $x = (C_3 T^H/2) - (1/C_3 T^H)(\xi_\alpha^2/2)$ and if $T > (\sqrt{2}\xi_\alpha / C_3)$ we obtain (4.3). The estimate (4.4) is obtained similarly. □

COROLLARY 4.1.

$$\lim_{T\to\infty} \sup_{\mu} I\!P_{\mu,\sigma}\{K_0^*\} = 0$$

and

$$\lim_{T\to\infty} \sup_{\mu} I\!P_{\mu,\sigma}\{K_0^*\} = 0.$$

Assume that we observe the process X at points $0 \leqslant t_{n,1} < \cdots < t_{n,n} \leqslant T$, where $t_{n,k} \in \pi^n$. Put $\Delta^n = \max\{t_{n,1}, |\pi^n|, T - t_{n,n}\}$ and assume that

$$\lim_{n\to\infty} \Delta^n = 0. \tag{4.6}$$

We will introduce a discrete version of the functional $l(X_\cdot)$ Put $s_k = t_{n,k}$, $\Delta s_k = s_{k+1} - s_k$, $x_k = X_{t_{n,k}}$ and $\Delta x_k = x_{k+1} - x_k$. Assume that $\widehat{\sigma}_n^2$ is some consistent estimator of σ^2. Put

$$\begin{aligned} l_n(x_1, \ldots, x_n) \;\doteq\; \exp\Bigg[& - C_0 \alpha_1 T^{2H-1} \sum_{k=0}^{n-1} s_{k+1}^{1/2H} (T - s_k)^{1/2-H} \Delta x_k + \\ & + C_0 \alpha_1 \sum_{k=1}^{n} s_{k+1}^{2H-2} \left(\sum_{i=0}^{k-1} s_{i+1}^{1/2-H} (s_k - s_i)^{1/2-H} \Delta x_i \right) \Delta s_k - \\ & - \beta_1 \widehat{\sigma}_n^2 B_2^2 T^{2H} \Bigg]. \end{aligned}$$

With the help of constants K_α^* and k_α^* from (4.1) and (4.2) define the critical domains

$$K_{1n}^* \doteq \{(x_{n,1}, \ldots, x_{n,n}) \in I\!R^n | l_n(x_{n,1}, \ldots, x_{n,n}) \geqslant K_\alpha^*\}$$

and

$$K_{2n}^* \doteq \{(x_{n,1},\ldots,x_{n,n}) \in I\!R^n | l_n(x_{n,1},\ldots,x_{n,n}) < k_\alpha^*\}.$$

If the observations belong to K_{1n}^* then **A** is accepted and if the observations belong to K_{2n}^* then **H** is accepted.

THEOREM 4.2. *Assume that we have* (4.6) *as* $n \to \infty$. *Then for any* $\mu \in I\!R$, $\sigma > 0$ *we have that*

$$l_n(x_{n,1},\ldots,x_{n,n}) \overset{I\!P_{\mu,\sigma,\sigma}}{\longrightarrow} l(X.|\sigma) \tag{4.7}$$

and

$$l_n(x_{n,1},\ldots,x_{n,n}) \overset{I\!P_{\mu,\sigma}}{\longrightarrow} l(X.|\sigma). \tag{4.8}$$

Proof. We prove the claim (4.7) (the claim (4.8) is proved similarly). Denote by $l(\widehat{X}.|\sigma)$ the random variable $l(X.|\sigma)$, when the process X_t is replaced by the process $\sigma Z_t + \mu t - (\sigma^2 t^{2H}/2)$, $0 \leqslant t \leqslant T$, and by $l_n(\widehat{x}_{n,1},\ldots,\widehat{x}_{n,n})$ the variable, where we replace $\Delta X_{n,k}$ by $\widehat{\sigma}_n \Delta Z_{n,k} + \mu\Delta t_k - (\sigma^2(\Delta t_k)^{2H}/2)$. Then for any $\epsilon > 0$, $C > 0$ we have that

$$\begin{aligned} &I\!P_{\mu,\sigma,\sigma}\{|l_n(x_{n,1},\ldots,x_{n,n}) - l(X.|\sigma)| > \epsilon\} \\ &\quad \leqslant I\!P_{\mu,\sigma,\sigma}\{|l_n(x_{n,1},\ldots,x_{n,n})| \geqslant C\} + I\!P_{\mu,\sigma,\sigma}\{|l(X.|\sigma)| \geqslant C\} + \\ &\quad + I\!P_{\mu,\sigma,\sigma}\left\{|\log l_n(\widehat{x}_{n,1},\ldots,\widehat{x}_{n,n}) - \log l(\widehat{X}.|\sigma)| > \frac{\epsilon}{\exp(C)}\right\}. \end{aligned} \tag{4.9}$$

The first two probabilities can be chosen sufficiently small for large $C > 0$. From the structure of the functionals $l(\widehat{X}.|\sigma)$ and $l_n(\widehat{x}_{n,1},\ldots,\widehat{x}_{n,n})$, the facts that $\widehat{\sigma}_n \overset{I\!P_{\mu,\sigma,\sigma}}{\longrightarrow} \sigma$,

$$C_0 \sum_{k=1}^{n-1} s_{k+1}^{1/2-H}(T-s_k)^{1/2-H}\Delta s_k \to \int_0^t z(t,s)\,\mathrm{d}s$$

and

$$\begin{aligned} &C_0 \sum_{k=1}^{n} s_k^{2H-2}\left(\sum_{i=1}^{k-1} s_{i+1}^{1/2-H}(s_k-s_i)^{1/2-H}\Delta s_i\right)\Delta s_k \\ &\quad \to \int_0^T s^{2H-2}\left(\int_0^s z(s,u)\,\mathrm{d}u\right)ds, \end{aligned}$$

it is sufficient to prove that

$$C_0 \sum_{k=1}^{n} s_{k+1}^{1/2-H}(T-s_k)^{1/2-H}\Delta Z_k \overset{I\!P_{\mu,\sigma,\sigma}}{\longrightarrow} M_T, \tag{4.10}$$

where

$$\triangle Y_k \doteq Y_{(k+1)T/n} - Y_{kT/n}$$

for any process Y, and that

$$C_0 \sum_{k=1}^{n} s_{k+1}^{2H-2} \left(\sum_{i=1}^{k-1} s_{i+1}^{1/2-H} (s_k - s_i)^{1/2-H} \Delta Z_i \right) \Delta s_k \xrightarrow{I\!P_{\mu,\sigma,\sigma}} \int_0^T s^{2H-2} M_s \, \mathrm{d}s. \tag{4.11}$$

To prove (4.10) consider

$$\begin{aligned} &I\!E\left(M_T - C_0 \sum_{k=1}^{n-1} s_{k+1}^{1/2-H} (T - s_k)^{1/2-H} \Delta Z_k \right)^2 \\ &\quad = H(2H-1) \int_0^T \int_0^T (z(T,s) - f_{n,T}(s)) \times \\ &\quad \times (z(T,u) - f_{n,T}(u)) |u - s|^{2H-2} \, \mathrm{d}u \, \mathrm{d}s, \end{aligned}$$

where $f_{n,T}(s) = C_0 s_{k+1}^{1/2-H} (T - s_k)^{1/2-H} 1_{\{s \in [s_k, s_{k+1})\}}$. We have that $f_{n,T}(s) \to z(T,s)$ for $s \in (0,T)$, $z(T,s) \geqslant f_{n,T}(s)$ and $\int_0^T \int_0^T z(T,s) z(T,u) |u - s|^{2H-2} \mathrm{d}u \, \mathrm{d}s < \infty$. Therefore, by monotone convergence,

$$\int_0^T \int_0^T (z(T,s) - f_{n,T}(s))(z(T,u) - f_{n,T}(u)) |u - s|^{2H-2} \, \mathrm{d}u \, \mathrm{d}s \to 0$$

as $n \to \infty$ and (4.10) follows.

To finish, we prove (4.11). Denote $g(s) \doteq s^{2H-2} M_s$ and

$$g_n(s) \doteq C_0 s_{k+1}^{2H-2} \sum_{i=1}^{k-1} s_{i+1}^{1/2-H} (s_k - s_i)^{1/2-H} \Delta Z_i 1_{\{s \in [s_k, s_{k+1})\}}.$$

Then, for any $s \in (0, T]$,

$$\begin{aligned} I\!E|g(s) - g_n(s)| \leqslant{} & |s^{2H-2} - s_{k+1}^{2H-2}| \, I\!E|M_s| + s^{2H-2} H(2H-1) \times \\ & \times \left(I\!E \int_0^s \int_0^s (z(s,u) - f_{n,s}(u))(z(s,r) - \right. \\ & \left. - f_{n,s}(r)) |u - r|^{2H-2} \, \mathrm{d}u \, \mathrm{d}r \right)^{1/2}, \end{aligned} \tag{4.12}$$

and as in previous inequalities, the second term on the right hand side of (4.12) goes to zero, which means that the left hand side can be dominated, according to (Mémin et al., 2001, (2.4)) by

$$C_H^1 s^{2H-2} T^{1-H} + C_H^2 ||z(s, \cdot)||_{L^{1/H}[0,s]} \leqslant C_H^3 T^{1-H},$$

where C_H^i are some constants, $i = 1, 2, 3$. From here

$$\mathbb{E}\left| \int_0^T (g(s) - g_n(s))\,\mathrm{d}s \right| \to 0,$$

as $n \to \infty$ and we obtain (4.11). □

COROLLARY 4.2. *Assume that* (4.6) *holds. Then*

$$\limsup_n \mathbb{P}_{\mu,\sigma}(K_{1n}^*) \leqslant \alpha, \qquad \limsup_n \mathbb{P}_{\mu,\sigma,\sigma}(K_{2n}^*) = 0$$

and

$$\lim_T \limsup_n (\mathbb{P}_{\mu,\sigma} + \mathbb{P}_{\mu,\sigma,\sigma})(K_{0n}^*) = 0,$$

where $K_{0n}^* \doteq \{(x_{n,1}, \ldots, x_{n,n}) : k_\alpha^* < \log l_n(x_{n,1}, \ldots, x_{n,n}) < K_\alpha^*\}$.

Proof. By Theorem 4.2 we have, as $n \to \infty$:

$$\mathbb{P}_{\mu,\sigma}(K_{1n}^*) \to \mathbb{P}_{\mu,\sigma}(K_1^*),\ \mathbb{P}_{\mu,\sigma,\sigma}(K_{2n}^*) \to \mathbb{P}_{\mu,\sigma}(K_2^*)$$

and

$$(\mathbb{P}_{\mu,\sigma} + \mathbb{P}_{\mu,\sigma,\sigma})(K_{0n}^*) \to (\mathbb{P}_{\mu,\sigma} + \mathbb{P}_{\mu,\sigma,\sigma})(K_0^*).$$

Hence the statements of the corollary follow from Theorem 4.2. □

Note that according to Corollary 4.2 the proposed test procedure has asymptotically the level of errors less or equal to α for both kind of errors. Note also that the probability not to make decision goes to zero as $T \to \infty$. It is also easy to see from the proof of Theorem 4.2 and Corollary 4.2 that this convergence is uniform for all μ and all $\sigma \geqslant \sigma_0 > 0$, where σ_0 is fixed.

5. Goodness-of-fit Test

5.1. INTRODUCTION

Suppose that **A** against **H** was tested, and we conclude that, for example, **H** is true. Consider certain functional depending on the trajectory of the observed process X_t, $0 \leqslant t \leqslant T$. If the distribution of this functional under **H** is known we can construct the corresponding goodness-of-fit test. For a given confidence level we either reject **H** or do not reject **H**. If we reject **H** it means that the observed trajectory does not fit the model described by **H**, and we conclude finally in this case that both **A** and **H** are wrong.

If the parameters in the models are unknown we propose an asymptotic test which provides a given confidence level as $T \to +\infty$.

5.2. THE WHOLE TRAJECTORY IS OBSERVED AND THE PARAMETERS μ AND σ ARE KNOWN

Introduce a functional which depends on the whole observed trajectory $X(t), t \in [0, T]$, in a linear way

$$Q_T \doteq \int_0^T z_1(T, s)\,\mathrm{d}X_s,$$

where

$$z_1(T, s) = s^{1/4-H}(T-s)^{3/4-H}.$$

We choose here the exponents $1/4 - H$ and $3/4 - H$ different from $1/2 - H$ in order to obtain the functional which is essentially different from (1.5). The reason of that will be clear from the Theorem 6.1. The integral exists in both cases when $X_t = \sigma Z_t + \mu t$ and $X_t = \sigma Z_t + \mu t - (\sigma^2/2)t^{2H}$.

Denote

$$B_4 = B\left(\tfrac{5}{4} - H, \tfrac{7}{4} - H\right), \qquad B_5 = B\left(H + \tfrac{1}{4}, \tfrac{7}{4} - H\right).$$

THEOREM 5.1. *Let the parameters μ and σ be known.*

- *Assume that we have* **A**: $X_t = \sigma Z_t + \mu t - (\sigma^2/2)t^{2H}$. *Then*
  $R_T^{\mathbf{A}} \doteq T^{H-1}Q_T - \mu B_4 \cdot T^{1-H} + \sigma^2 H \cdot B_5 \cdot T^H \sim N(0, C_H^3\sigma^2)$;
- *Assume that we have* **H**: $X_t = \sigma Z_t + \mu t$. *Then*
  $R_T^{\mathbf{H}} \doteq T^{H-1}Q_T - \mu B_4 \cdot T^{1-H} \sim N(0, C_H^3\sigma^2)$,

where

$$C_H^3 \doteq H(2H-1)\int_0^1\int_0^1 (us)^{1/4-H}((1-u)(1-s))^{3/4-H} \cdot \cdot|u-s|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s.$$

Proof. Assume **A**. Then we have

$$Q_T = \sigma\int_0^T z_1(T, s)\,\mathrm{d}Z_s + \mu T^{2-2H}B_4 - \sigma^2 H B_5 T \tag{5.1}$$

and so

$$R_T^{\mathbf{A}} = T^{H-1}\sigma\int_0^T z_1(T, s)\,\mathrm{d}Z_s.$$

Obviously, $R_T^{\mathbf{A}}$ is normally distributed with mean zero and with variance

$$\mathbb{E}(R_T^{\mathbf{A}})^2 = \sigma^2 T^{2H-2}H(2H-1)\int_0^T\int_0^T (us)^{1/4-H} \times \times ((T-u)(T-s))^{3/4-H}|s-u|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s,$$

that is, $\mathbb{E}(R_T^{\mathbf{A}})^2 = \sigma^2 C_H^3$ and the first claim now follows.

Assume **H**. Then we can write Q_T as

$$Q_T = \sigma \int_0^T z_1(T,s)\,\mathrm{d}Z_s + \mu T^{2-2H} B_4, \tag{5.2}$$

and the second claim follows from (5.2) as above.

The goodness-of-fit tests are based on the statistics

$$\overline{R_T^{\mathbf{A}}} \doteq \frac{R_T^{\mathbf{A}}}{\sigma (C_H^3)^{1/2}}, \qquad \overline{R_T^{\mathbf{H}}} \doteq \frac{R_T^{\mathbf{H}}}{\sigma (C_H^3)^{1/2}}.$$

Fix a confidence level $1-\alpha, \alpha \in (0, 1/2)$, and $\xi_{\alpha/2}$ be an $(\alpha/2)$- quantile of a standard normal law, that is, $P\{N(0,1) \geqslant \xi_{\alpha/2}\} = \alpha/2$. We reject **A** if $|\overline{R_T^{\mathbf{A}}}| > \xi_{\alpha/2}$, and reject **H** if $|\overline{R_T^{\mathbf{H}}}| > \xi_{\alpha/2}$.

Note that under **A**, $\overline{R_T^{\mathbf{H}}} \xrightarrow{I\!P_{\mu,\sigma,\sigma}} -\infty, T \to +\infty$, therefore the inequality $\overline{R_T^{\mathbf{H}}} < -\xi_{\alpha/2}$ is an additional argument in favor of **A**.

Also, if **H** is true, then $\overline{R_T^{\mathbf{A}}} \xrightarrow{I\!P_{\mu,\sigma}} +\infty, T \to +\infty$, therefore the inequality $\overline{R_T^{\mathbf{A}}} > \xi_{\alpha/2}$ is an additional argument in favor of **H**. □

Remark 5.1. Suppose that in reality we have the model $X_t = \sigma Z_t^{H_1} + \mu t$, $H_1 > H$, not $\sigma Z_t^H + \mu t$. Denote the law of X in this case by $I\!P$. Then

$$\overline{R_T^{\mathbf{H}}} = \frac{T^{H-1}}{(C_H^3)^{1/2}} \int_0^T s^{1/4-H}(T-s)^{3/4-H}\,\mathrm{d}Z_s^{H_1},$$

and $I\!E(\overline{R_T^{\mathbf{H}}})^2$ has the order $T^{2(H_1-H)}$ for large T, thus $\overline{R_T^{\mathbf{H}}} \xrightarrow{I\!P} \infty, T \to \infty$. And

$$\overline{R_T^{\mathbf{A}}} = \overline{R_T^{\mathbf{H}}} + \frac{\sigma H B_5 T^H}{(C_H^3)^{1/2}} = T^{H_1-H} O_{I\!P}(1) + \frac{\sigma H B_5 T^H}{(C_H^3)^{1/2}} \xrightarrow{I\!P} +\infty, T \to \infty.$$

Therefore our statistics can distinguish this case, too.

6. Goodness-of-fit Tests with Discrete Observations

6.1. ASYMPTOTIC BEHAVIOR OF DISCRETE STATISTICS FOR μ UNKNOWN AND σ KNOWN

Suppose for the simplicity that we observe the values $X_{kT/n}, k = 0, 1, \ldots, n$. We substitute in $R_T^{\mathbf{A}}, R_T^{\mathbf{H}}$ a discretization of Q_T,

$$\widehat{Q_T} \doteq \sum_{k=0}^{n-1} \left(\frac{(k+1)T}{n}\right)^{1/4-H} \left(T - \frac{kT}{n}\right)^{3/4-H} \triangle X_k,$$

Instead of μ we substitute the estimates (3.2) and (3.3), respectively. Thus we define

$$\widehat{R_T^{\mathbf{A}}} \doteq T^{H-1}\widehat{Q_T} - \widehat{\mu_A} B_4 T^{1-H} + \sigma^2 H B_5 T^H$$

and

$$\widehat{R_T^{\mathbf{H}}} \doteq T^{H-1}\widehat{Q_T} - \widehat{\mu_H} B_4 T^{1-H}.$$

Under the hypothesis **A** we have

$$\begin{aligned}\widehat{R_T^{\mathbf{A}}} &= \sigma T^{-H} \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^{1/4-H} \left(1-\frac{k}{n}\right)^{3/4-H} \triangle Z_{kT/n} + \\ &\quad + \mu T^{1-H} \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^{1/4-H} \left(1-\frac{k}{n}\right)^{3/4-H} \cdot \frac{1}{n} - \\ &\quad - \frac{\sigma^2}{2} T^H \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^{1/4-H} \left(1-\frac{k}{n}\right)^{3/4-H} \times \\ &\quad \times \left(\left(\frac{k+1}{n}\right)^{2H} - \left(\frac{k}{n}\right)^{2H}\right) - \widehat{\mu_A} B_4 T^{1-H} + \sigma^2 H B_5 T^H. \qquad (6.1)\end{aligned}$$

And under the hypothesis **H**

$$\begin{aligned}\widehat{R_T^{\mathbf{H}}} &= \sigma T^{-H} \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^{1/4-H} \left(1-\frac{k}{n}\right)^{3/4-H} \triangle Z_{kT/n} + \\ &\quad + \mu T^{1-H} \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^{1/4-H} \left(1-\frac{k}{n}\right)^{3/4-H} \cdot \frac{1}{n} - \widehat{\mu_H} B_4 T^{1-H}. \qquad (6.2)\end{aligned}$$

For the beginning we find the rate of convergence of the integral sums in (6.1) and (6.2) to the corresponding integrals.

Define $\widetilde{R_T^{\mathbf{H}}}$ by

$$\widetilde{R_T^{\mathbf{H}}} \doteq \frac{\sigma}{T^{1-H}} \int_0^T s^{1/4-H} (T-s)^{3/4-H}\, \mathrm{d}Z_s + B_3 T^{1-H} (\mu - \widehat{\mu_H})$$

and $\widetilde{R_T^{\mathbf{A}}}$ similarly, with $\widehat{\mu_A}$ replacing $\widehat{\mu_H}$.

We study the differences $\widehat{R_T^{\mathbf{A}}} - \widetilde{R_T^{\mathbf{A}}}$ and $\widehat{R_T^{\mathbf{H}}} - \widetilde{R_T^{\mathbf{H}}}$. Put

$$q_n(T,s) \doteq \sum_{k=0}^{n-1} \left(\frac{(k+1)T}{n}\right)^{1/4-H} \left(T - \frac{kT}{n}\right)^{3/4-H} \cdot 1_{[kT/n,(k+1)T/n)}(s),$$

$$I(\alpha,\beta) = \int_0^1 s^\alpha (1-s)^\beta\, \mathrm{d}s, \qquad I_n(\alpha,\beta) = \sum_{k=0}^{n-1} \left(\frac{k+1}{n}\right)^\alpha \left(1-\frac{k}{n}\right)^\beta \frac{1}{n},$$

for $n \geqslant 2$.

We have that

$$\begin{aligned}\widehat{R_T^{\mathbf{H}}} - \widetilde{R_T^{\mathbf{H}}} &= T^{H-1} \int_0^T (q_n(T,s) - q(T,s))\, \mathrm{d}Z_s - T^{1-H} \mu (I_n(1/4 - \\ &\quad - H, 3/4 - H) - I(1/4 - H, 3/4 - H)) \qquad (6.3)\end{aligned}$$

and

$$\widehat{R_T^{\mathbf{A}}} - \widetilde{R_T^{\mathbf{A}}} = \widehat{R_T^{\mathbf{H}}} - \widetilde{R_T^{\mathbf{H}}} - \frac{\sigma^2}{2}T^H 2H\Bigg(I_n\Big(H-\frac{3}{4},\frac{3}{4}-H\Big) - I\Big(H-\frac{3}{4},\frac{3}{4}-H\Big)\Bigg) - \frac{\sigma^2}{2}T^H\Bigg(\sum_{k=0}^{n-1}\Bigg(\Big(\frac{k+1}{n}\Big)^{1/4-H} \times \Big(1-\frac{k}{n}\Big)^{3/4-H}\Bigg(\Big(\frac{k+1}{n}\Big)^{2H} - \Big(\frac{k}{n}\Big)^{2H}\Bigg) - 2H\Big(\frac{k+1}{n}\Big)^{H-3/4}\Big(1-\frac{k}{n}\Big)^{3/4-H}\frac{1}{n}\Bigg)\Bigg) \tag{6.4}$$

Using self-similarity,

$$\mathbb{E}\Bigg(T^{H-1}\int_0^T (q_n(T,s) - q(T,s))\,\mathrm{d}Z_s\Bigg)^2 = \mathbb{E}\Bigg(\int_0^1 (q_n(1,s) - q(1,s))\,\mathrm{d}Z_s\Bigg)^2. \tag{6.5}$$

Accordingly to (Mémin et al., 2001, Theorem 1.1) we have

$$\mathbb{E}\Bigg(\int_0^1 (q_n(1,s) - q(1,s))\,\mathrm{d}Z_s\Bigg)^2 \leqslant c_H ||q_n(1,s) - q(1,s)||^2_{L^{1/H}(0,1)}. \tag{6.6}$$

Now we use these preliminary calculations to prove the next result. Let $n = n(T)$ be the number of approximation points.

THEOREM 6.1. *Assume*

(a) *For* $1/2 < H \leqslant 3/4$,
$$\frac{T^\beta}{n(T)} \to 0, \quad T \to \infty, \quad \textit{with } \beta = \frac{H}{H+1/4}.$$
(b) *For* $3/4 < H < 1$,
$$\frac{T^\beta}{n(T)} \to 0, \quad T \to \infty, \quad \textit{with } \beta = H.$$

Then under **A**

$$\widehat{R_T^{\mathbf{A}}} - \widetilde{R_T^{\mathbf{A}}} = o_{\mathbb{P}}(1), \quad T \to \infty \tag{6.7}$$

and under **H**

$$\widehat{R_T^{\mathbf{H}}} - \widetilde{R_T^{\mathbf{H}}} = o_{\mathbb{P}}(1), \quad T \to \infty. \tag{6.8}$$

Moreover under **A** $\widetilde{R_T^{\mathbf{A}}} \sim N\left(0,\rho^2\right)$, *and* **H** $\widetilde{R_T^{\mathbf{H}}} \sim N\left(0,\rho^2\right)$, *where*

$$\rho^2 \doteq \sigma^2 H(2H-1)\int_0^1\int_0^1 \varphi(s)\varphi(u)\cdot|u-s|^{2H-2}\,\mathrm{d}u\,\mathrm{d}s,$$

with

$$\varphi(s) \doteq s^{1/4-H}\,(1-s)^{3/4-H} - \frac{B_3}{B_1}s^{1/2-H}\,(1-s)^{1/2-H}\,.$$

Proof. To prove the claims note first that using Lemmas A.1 and B.1 we have that $\widehat{R_T^{\mathbf{H}}} - \widetilde{R_T^{\mathbf{H}}} = o_{I\!P}(1)$ under **H** and $\widehat{R_T^{\mathbf{A}}} - \widetilde{R_T^{\mathbf{A}}} = o_{I\!P}(1)$ under **A**.

Next, we substitute (3.2) into $\widetilde{R_T^{\mathbf{A}}}$ and obtain

$$\widetilde{R_T^{\mathbf{A}}} = \frac{\sigma}{T^{1-H}} \int_0^T \left[s^{1/4-H}(T-s)^{3/4-H} - \frac{B_3}{B_1}s^{1/2-H}(T-s)^{1/2-H} \right] \mathrm{d}Z_s.$$

This implies that under **A** $\widetilde{R_T^{\mathbf{A}}} \sim N(0,\rho^2)$. Similarly, one shows that under **H** $\widetilde{R_T^{\mathbf{H}}} \sim N\left(0,\rho^2\right)$. □

Remark 6.1. For the kernel $z(t,s)$ instead of $z_1(t,s)$ we obtain the degenerate distribution of $\widetilde{R_T^{\mathbf{A}}}$ and $\widetilde{R_T^{\mathbf{H}}}$. This is the reason why we take the kernel $z_1(t,s)$.

6.2. GOODNESS-OF-FIT TEST

Based on Theorem 6.1, we construct the goodness-of-fit test similarly to the one from Section 5.2. Choose $\xi_{\alpha/2}$ as there. We reject **A** if $|\widehat{R_T^{\mathbf{A}}}| > \rho\xi_{\alpha/2}$, and we reject **H** if $|\widehat{R_T^{\mathbf{H}}}| > \rho\xi_{\alpha/2}$. The test is applicable for large T only, contrary to the test from Section 5.2, because for the probability $\mathrm{e}_{\mathbf{A}}(T)$ that **A** is rejected when **A** is true, we have now

$$\lim_{T\to\infty} \mathrm{e}_{\mathbf{A}}(T) = \alpha$$

and similarly for **H** and $\mathrm{e}_{\mathbf{H}}(T)$.

7. On Volatility Estimation

In this section we construct an estimator for the parameter σ. We end this section by giving goodness-of-fit test for the case where both μ and σ are unknown.

7.1. INTRODUCTORY COMPUTATIONS FOR VOLATILITY ESTIMATION

Assume **A**. Then the background process is $X_t = \sigma Z_t + \mu t - \sigma^2/2t^{2H}$, $t \geqslant 0$. We make observations at time points $t_k = kT/n$, $k = 0, 1, \ldots, n$. We have that

$$\Delta X_k = \sigma\Delta Z_k + \mu\Delta t_k - \tfrac{1}{2}\sigma^2\Delta(t^{2H})_k, \quad k = 0, \ldots, n-1.$$

Consider now $\Delta X_k/T^H$ and write this as

$$\frac{\Delta X_k}{T^H} = \sigma\frac{1}{n^H}\varepsilon_k + \frac{\mu\Delta t_k}{T^H} - \frac{\sigma^2}{2T^H}\Delta(t^{2H})_k. \tag{7.1}$$

In (7.1) we used the notation $\varepsilon_k = \Delta Z_k n^H / T^H$. By self-similarity the distribution of the vector $(\varepsilon_0, \ldots, \varepsilon_{n-1})$ is the same as of the vector

$$\left(\frac{Z_{1/n} - Z_0}{1/n^H}, \ldots, \frac{Z_1 - Z_{(n-1)/n}}{1/n^H}\right) \stackrel{d}{=} (Z_1 - Z_0, Z_2 - Z_1, \ldots, Z_n - Z_{n-1}),$$

where we again used self-similarity. Simple computation gives $\mathbb{E}\varepsilon_k = 0$, $\mathbb{E}\varepsilon_k^2 = 1$ and

$$\mathbb{E}\varepsilon_k \varepsilon_l = \tfrac{1}{2}(|k-l+1|^{2H} - 2|k-l|^{2H} + |k-l-1|^{2H}).$$

If $k > l \geqslant 1$ and $1/2 \leqslant H < 1$, then, applying the mean value theorem twice gives

$$0 \leqslant E\varepsilon_k \varepsilon_l \leqslant H(2H-1)(k-l)^{2H-2}. \tag{7.2}$$

Denote $\mu_1 \doteq n^H \mu \Delta t / T^H$, $y_t \doteq n^H \Delta X_t / T^H$ and rewrite (7.1):

$$y_t = \sigma \varepsilon_t + \mu_1 - \tfrac{1}{2}\sigma^2 T^{-H} n^H \Delta t^{2H}.$$

To simplify the notation put

$$y_k \doteq \sigma \varepsilon_k + \mu_1 - \tfrac{1}{2}\sigma^2 T^H n^H \Delta\tau_k^{2H}, \quad k = 0, 1, \ldots, n-1, \tag{7.3}$$

where $\triangle\tau_k^{2H} \doteq (k + 1/n)^{2H} - (k/n)^{2H}$. We use a sample variance to estimate σ:

$$\widehat{\sigma}_n^2 \doteq \frac{n}{n-1}(\overline{y_n^2} - (\overline{y_n})^2) \quad \text{with } \overline{y_n} \doteq \frac{y_1 + \cdots + y_n}{n}. \tag{7.4}$$

Let $z_k \doteq \sigma \varepsilon_k - (\sigma^2/2) T^H n^H \Delta\tau_k^{2H}$, $\quad k = 0, 1, \ldots, n-1$. Then

$$\overline{z}_n = \sigma \overline{\varepsilon}_n - \tfrac{1}{2}\sigma^2 T^H n^{H-1} \tag{7.5}$$

and

$$\begin{aligned}
\widehat{\sigma}^2 &= \frac{n}{n-1}\left(\overline{z_n^2} - (\overline{z_n})^2\right) \\
&= \frac{n\sigma^2}{n-1}\Bigg(\overline{\varepsilon_n^2} - \sigma T^H n^H \overline{\varepsilon_n \Delta\tau_n^{2H}} + \frac{\sigma^2}{4} T^{2H} n^{2H} \overline{\left(\Delta\tau_n^{2H}\right)^2} - (\overline{\varepsilon_n})^2 + \\
&\quad + \sigma T^H n^H \overline{\varepsilon_n}\,\overline{\Delta\tau_n^{2H}} - \frac{\sigma^2}{4} T^{2H} n^{2H} \left(\overline{\Delta\tau_n^{2H}}\right)^2 \Bigg).
\end{aligned} \tag{7.6}$$

Again we have a problem with the rate of the discretization with respect to the observation interval. We start with obvious lemma:

LEMMA 7.1. *Assume that X, Y are two standard normal random variables*:

$$\mathbb{E}X = \mathbb{E}Y = 0 \quad \text{and} \quad \mathrm{Var}(X) = \mathrm{Var}(Y) = 1.$$

Assume that $\mathbb{E}XY = r$. Then

$$\mathbb{E}((X^2 - 1)(Y^2 - 1)) = 2r^2. \tag{7.7}$$

LEMMA 7.2. *With the notation above*:

- *If* $H < 3/4$, *then* $\mathbb{E}|\overline{\epsilon_n^2} - 1| \leqslant C/\sqrt{n}$.
- *If* $H = 3/4$, *then* $\mathbb{E}|\overline{\epsilon_n^2} - 1| \leqslant C\sqrt{\log n/n}$.
- *If* $3/4 < H < 1$, *then* $\mathbb{E}|\overline{\epsilon_n^2} - 1| \leqslant Cn^{2H-2}$.

Proof. We have that

$$\overline{\epsilon_n^2} - 1 = \frac{1}{n}\sum_{i=0}^{n-1}(\epsilon_i^2 - 1).$$

From Lemma 7.1 and (7.2):

$$\begin{aligned}\mathbb{E}(\overline{\epsilon_n^2} - 1)^2 &= \frac{1}{n^2}\sum_{i=0}^{n-1}\mathbb{E}(\epsilon_i^2 - 1)^2 + \frac{2}{n^2}\sum_{0\leqslant j<i\leqslant n-1}\mathbb{E}(\epsilon_i^2 - 1)(\epsilon_j^2 - 1)\\ &\leqslant \frac{C}{n} + \frac{C}{n^2}\sum_{0\leqslant j<i\leqslant n-1}(i-j)^{4H-4}. \qquad (7.8)\end{aligned}$$

Note that

$$\sum_{0\leqslant j<i\leqslant n-1}(i-j)^{4H-4} = \sum_{j=1}^{n-1}(n-j)j^{4H-4}.$$

This and the inequality (7.8) give the result.

We have

$$\overline{z_n} \stackrel{d}{=} \sigma n^{H-1}(Z_1 - \tfrac{1}{2}\sigma T^H) \qquad (7.9)$$

so that

$$0 \leqslant \mathbb{E}\overline{z_n}^2 \leqslant \sigma^2 n^{2H-2}(2\mathbb{E}Z_1^2 + \tfrac{1}{2}\sigma^2 T^{2H}). \qquad (7.10)$$

7.2. ESTIMATION OF σ

THEOREM 7.1. *Assume* **A**. *If* $n(T)$ *is such that* $T^{3H}/n(T)^{2-2H} \to 0$, *then* $T^H(\widehat{\sigma}_n^2 - \sigma^2) = o_{\mathbb{P}}(1)$.

Assume **H**. *Then*

(a) *If* $1/2 < H < 3/4$ *and* $n(T)$ *is such that* $T^{2H}/n(T) \to 0$, *then* $T^H(\widehat{\sigma}_n^2 - \sigma^2) = o_{\mathbb{P}}(1)$.
(b) *If* $H = 3/4$ *and* $n(T)$ *is such that* $T^{3/2}\log(n(T))/n(T) \to 0$, *then* $T^H(\widehat{\sigma}_n^2 - \sigma^2) = o_{\mathbb{P}}(1)$.
(c) *If* $3/4 < H < 1$ *and* $n(T)$ *is such that* $T^H/n(T)^{2-2H} \to 0$, *then* $T^H(\widehat{\sigma}_n^2 - \sigma^2) = o_{\mathbb{P}}(1)$.

Proof. Using Lemma B.2 and (7.6)

$$T^H\mathbb{E}|\widehat{\sigma}_n^2 - \sigma^2| \leqslant C\left(T^H\mathbb{E}|\overline{\epsilon}_n^2 - 1| + \frac{T^H + T^{2H} + T^{3H}}{n^{2-2H}}\right).$$

where C depends on σ^2. Under **A** the statement follows from Lemma 7.2. Under **H** we have

$$\widehat{\sigma}_n^2 = \frac{n\sigma^2}{n-1}(\overline{\epsilon_n^2} - (\overline{\epsilon_n})^2),$$

and

$$T^H \cdot E|\widehat{\sigma}_n^2 - \sigma^2| \leqslant C\left(T^H I\!E|\overline{\epsilon}_n^2 - 1| + \frac{T^H}{n^{2-2H}}\right).$$

The claims (a)–(c) follow from Lemma 7.2.

7.3. GOODNESS-OF-FIT TEST WITH UNKNOWN μ AND σ

If the parameter σ is unknown, then using the observation $X_{kT/n}$, $k = 0, 1, \ldots, n$, with $n = n(T)$, an estimator $\widehat{\sigma^2} \doteq \widehat{\sigma_n}^2$ is constructed. The construction of this estimator is explained in Section 7.2.

If

$$\frac{T^{(3H/2-2H)}}{n(T)} \to 0, \quad T \to \infty \tag{7.11}$$

we have $(\widehat{\sigma^2} - \sigma^2) \cdot T^H \xrightarrow{I\!P_{\mu,\sigma,\sigma}} 0$, when **A** is true.

If the conditions (a)–(c) of Theorem 7.1 hold, we have the same convergence for $(\widehat{\sigma}^2 - \sigma^2) \cdot T^H$, then **H** is true.

Define the statistics

$$\widehat{S_T^A} \doteq \left.\frac{\widehat{R_T^A}}{\rho}\right|_{\sigma=\widehat{\sigma}}, \qquad \widehat{S_T^H} \doteq \left.\frac{\widehat{R_T^H}}{\rho}\right|_{\sigma=\widehat{\sigma}}. \tag{7.12}$$

Consider the model with unknown μ and σ.

THEOREM 7.2.

(1) *Assume that* **H** *is true and the conditions* (a)–(c) *of Theorem* 7.1 *hold. Then* $\widehat{S_T^{\mathbf{H}}} \to N(0, 1)$ *in distribution.*
(2) *Assume that* **A** *is true, and that* $T^{3H}/n(T)^{2-2H} \to 0$, $T \to \infty$. *Then* $\widehat{S_T^{\mathbf{A}}} \to N(0, 1)$ *in distribution.*

Proof. Suppose that **A** is true. By Theorem 7.1 we have $\widehat{\sigma^2} - \sigma T^H \xrightarrow{P} 0$. Rewrite (7.12) as

$$\begin{aligned}\widehat{S_T^{\mathbf{A}}} &= \frac{\widehat{R_T^{\mathbf{A}}}|_{\sigma=\widehat{\sigma}}}{\rho} \cdot \frac{\sigma}{\widehat{\sigma}} \\ &= \frac{\sigma}{\widehat{\sigma}}\left(\frac{\widehat{R_T^{\mathbf{A}}} + B_4 T^{1-H}(\widehat{\mu_{\mathbf{A}}} - \mu_{\mathbf{A}}|_{\sigma=\widehat{\sigma}}) + H B_5 T^H(\widehat{\sigma}^2 - \sigma^2)}{\rho}\right).\end{aligned}$$

Now, $\widehat{\sigma} \xrightarrow{\mathbb{P}} \sigma$ and $HB_5T^H(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{\mathbb{P}} 0$, as $T \to \infty$. From (3.2)

$$B_4T^{1-H}(\widehat{\mu_{\mathbf{A}}} - \mu_{\mathbf{A}}|_{\sigma=\widehat{\sigma}}) = (HB_2B_4/B_1)T^H(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{P} 0. \tag{7.13}$$

And now from (7.13) and Theorem 7.1 the convergence $\widehat{S_T^A} \to N(0,1)$ follows.

If $\mathbf{H}$ is true then $T^H(\widehat{\sigma}^2 - \sigma^2) \xrightarrow{\mathbb{P}} 0$ holds under the conditions of the Theorem 7.1. The proof now follows in the same line. □

The goodness-of-fit test organized as follows. We reject $\mathbf{A}$ if $|\widehat{S_T^{\mathbf{A}}}| > \xi_{\alpha/2}$, and we reject $\mathbf{H}$ if $|\widehat{S_T^{\mathbf{H}}}| > \xi_{\alpha/2}$. Asymptotic relations for the errors $\mathrm{e}_{\mathbf{A}}(T)$ and $\mathrm{e}_{\mathbf{H}}(T)$ are the same in Section 6.1.

Appendix A. Approximation of Beta Integrals

LEMMA A.1. *Assume that* $-1 < \alpha < 0$, $\beta > -1$ *and* $n \geqslant 2$. *Then for* $\beta \geqslant 0$

$$|I(\alpha,\beta) - I_n(\alpha,\beta)| \leqslant C_1(\alpha,\beta)n^{-\alpha-1}, \tag{A.1}$$

and with $-1 < \beta < 0$ *we have*

$$|I(\alpha,\beta) - I_n(\alpha,\beta)| \leqslant C_2(\alpha,\beta)n^{-\alpha-\beta-1} \tag{A.2}$$

[*for the value of the constants*, *see the proof*].

Proof. We start the proof with

$$\begin{aligned}
&I(\alpha,\beta) - I_n(\alpha,\beta)\\
&\quad= \int_0^{\frac{1}{n}} s^\alpha(1-s)^\beta \mathrm{d}s - n^{-\alpha-1} + \\
&\quad\quad+ \sum_{k=1}^{n-2}\int_{k/n}^{k+1/n}\left(s^\alpha(1-s)^\beta - \left(\frac{k+1}{n}\right)^\alpha\left(1-\frac{k}{n}\right)^\beta\right)\mathrm{d}s + \\
&\quad\quad+ \int_{1-1/n}^{1} s^\alpha(1-s)^\beta\,\mathrm{d}s - n^{-\beta-1}.
\end{aligned}$$

We work first with the integral on $(0, 1/n)$. We have

$$\begin{aligned}
\int_0^{1/n} s^\alpha(1-s)^\beta\,\mathrm{d}s - n^{-\alpha-1} &= \int_0^{1/n}(s^\alpha - n^{-\alpha})\,\mathrm{d}s + \\
&\quad+ \int_0^{1/n} s^\alpha((1-s)^\beta - 1)\,\mathrm{d}s;
\end{aligned} \tag{A.3}$$

here

$$0 \leqslant \int_0^{1/n}(s^\alpha - n^{-\alpha})\,\mathrm{d}s = -\alpha/(\alpha+1)n^{-\alpha-1},$$

if $\beta \geqslant 0$, then

$$\left|\int_0^{1/n} s^\alpha((1-s)^\beta - 1)\,\mathrm{d}s\right| \leqslant \int_0^{1/n} s^\alpha\,\mathrm{d}s$$

and if $\beta < 0$ and $s \leqslant 1/n$, then $0 \leqslant (1-s)^\beta - 1 \leqslant 2^{-\beta} - 1$. Use these estimates in (A.3) to obtain

$$\left| \int_0^{1/n} s^\alpha (1-s)^\beta \, \mathrm{d}s - n^{-\alpha-1} \right| \leqslant C_1(\alpha, \beta) n^{-\alpha-1}. \tag{A.4}$$

Next, we work with the integral on $(1-1/n, 1)$. We have

$$\begin{aligned} \int_{1-1/n}^1 s^\alpha (1-s)^\beta \, \mathrm{d}s - n^{-\beta-1} &= \int_{1-1/n}^1 ((1-s)^\beta - n^{-\beta}) \, \mathrm{d}s + \\ &\quad + \int_{1-1/n}^1 (1-s)^\beta (s^\alpha - 1) \, \mathrm{d}s \end{aligned}$$

and this gives

$$\left| \int_{1-1/n}^1 s^\alpha (1-s)^\beta \, \mathrm{d}s - n^{-\beta-1} \right| \leqslant \frac{|\beta|}{1+\beta} n^{-\beta-1} + 2^{-\alpha} n^{-\beta-1}. \tag{A.5}$$

We continue with the middle term. We have

$$\begin{aligned} &\sum_{k=1}^{n-2} \left(\int_{k/n}^{k+1/n} s^\alpha (1-s)^\beta \, \mathrm{d}s - \left(\frac{k+1}{n}\right)^\alpha \left(1 - \frac{k}{n}\right)^\beta \frac{1}{n} \right) \\ &\quad = \sum_{k=1}^{n-2} \left(\int_{k/n}^{k+1/n} \left(s^\alpha - \left(\frac{k+1}{n}\right)^\alpha \right) (1-s)^\beta \, \mathrm{d}s \right) + \\ &\qquad + \sum_{k=1}^{n-2} \left(\int_{k/n}^{k+1/n} \left(\frac{k+1}{n}\right)^\alpha \left((1-s)^\beta - \left(1 - \frac{k}{n}\right)^\beta \right) \mathrm{d}s \right). \end{aligned} \tag{A.6}$$

The first term on the right hand side of (A.6) is always positive, when $\alpha < 0$. We use the estimate

$$s^\alpha - \left(\frac{k+1}{n}\right)^\alpha \leqslant \left(\frac{k}{n}\right)^\alpha - \left(\frac{k+1}{n}\right)^\alpha.$$

If $\beta \geqslant 0$, then $(1-s)^\beta \leqslant 1$ and so for the first term on the right hand side of (A.6) we obtain

$$\begin{aligned} 0 &\leqslant \sum_{k=1}^{n-2} \left(\int_{k/n}^{(k+1)/n} \left(s^\alpha - \left(\frac{k+1}{n}\right)^\alpha \right) (1-s)^\beta \, \mathrm{d}s \right) \\ &\leqslant n^{-\alpha-1} \sum_{k=1}^{n-2} (k^\alpha - (k+1)^\alpha) \leqslant n^{-\alpha-1} \end{aligned} \tag{A.7}$$

If $\beta \leqslant 0$ then

$$\begin{aligned}&\int_{k/n}^{(k+1)/n}\left(s^{\alpha}-\left(\frac{k+1}{n}\right)^{\alpha}\right)(1-s)^{\beta}\,\mathrm{d}s\\&\quad\leqslant\frac{1}{1+\beta}n^{-\alpha-\beta-1}(k^{\alpha}-(k+1)^{\alpha})((n-k)^{\beta+1}-(n-(k+1))^{\beta+1})\\&\quad\leqslant n^{-\alpha-\beta-1}(k^{\alpha}-(k+1)^{\alpha})\end{aligned}$$

and this gives the bound

$$0\leqslant\sum_{k=1}^{n-2}\left(\int_{k/n}^{(k+1)/n}\left(s^{\alpha}-\left(\frac{k+1}{n}\right)^{\alpha}\right)(1-s)^{\beta}\,\mathrm{d}s\right)\leqslant n^{-\alpha-\beta-1}. \tag{A.8}$$

Finally, the second part of the middle term is

$$J_n\doteq\sum_{k=1}^{n-2}\left(\int_{k/n}^{(k+1)/n}\left(\frac{k+1}{n}\right)^{\alpha}((1-s)^{\beta}-(1-k/n)^{\beta})\,\mathrm{d}s\right).$$

If $\beta \geqslant 0$, then with calculations, similar to above

$$|J_n|\leqslant n^{-\alpha-1} \tag{A.9}$$

and if $\beta < 0$, then

$$|J_n|\leqslant-\frac{1}{\beta}2^{\beta}n^{-\alpha-\beta-1}. \tag{A.10}$$

Combining the bounds (A.17), (A.18), (A.20) and (A.22) we get $C_1(\alpha,\beta)$, and combining the bounds (A.17), (A.18), (A.21) and (A.23) we get $C_2(\alpha,\beta)$. □

B. Estimations for Kernels

LEMMA B.1. *Put*

$$\begin{aligned}H_n\ \doteq\ &\sum_{k=0}^{n-1}\left(\left(\frac{k+1}{n}\right)^{1/4-H}\left(1-\frac{k}{n}\right)^{3/4-H}\left(\left(\frac{k+1}{n}\right)^{2H}-\left(\frac{k}{n}\right)^{2H}\right)-\right.\\&\left.-2H\left(\frac{k+1}{n}\right)^{H-3/4}\left(1-\frac{k}{n}\right)^{3/4-H}\frac{1}{n}\right)\end{aligned}$$

Then

$$|H_n|\leqslant Cn^{-\min(1,1/4+H)}. \tag{B.11}$$

Proof. The proof of Lemma B.1 is similar to Lemma A.1 and we leave it to the reader. □

LEMMA B.2. *Consider the expression*

$$\overline{u}_n(H) \doteq \frac{1}{n}\sum_{k=0}^{n-1}\left(\left(\frac{k+1}{n}\right)^{2H} - \left(\frac{k}{n}\right)^{2H}\right)^2.$$

Then

$$|\overline{u}_n(H)| \leqslant \frac{C}{n^2}. \tag{B.12}$$

Proof. The proof of this lemma is obvious. □

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