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# ASYMPTOTIC PROPERTIES OF AN INTENSITY ESTIMATOR OF AN INHOMOGENEOUS POISSON PROCESS IN A COMBINED MODEL*

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**Abstract.** A stochastic process with a drift, a diffusion, and a Poisson component is considered, where the last is an inhomogeneous process with unknown intensity $\lambda = \lambda(t)$ belonging to a compact of a Sobolev space. By observations over the process within a time interval $[0, T]$ we construct the maximum likelihood estimator (MLE) of $\lambda$. Conditions providing consistency of the estimator and asymptotic normality of the functionals of it are studied. A comparison is given of the MLEs constructed by the observations over the whole process and over its individual components.

**Key words.** inhomogeneous Poisson process, intensity, drift, diffusion, maximal likelihood estimate, consistency, asymptotic normality, Sobolev space

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**1. Introduction.** The scheme we consider looks like this. A stochastic process on the nonnegative semi-axis is observed which has independent increments and consists of three components: a drift, a diffusion, and an inhomogeneous Poisson process with unknown intensity $\lambda = \lambda(t)$, $t \geqq 0$. By observations over the realizations of this process we construct the maximum likelihood estimator for $\lambda(t)$. The following papers, related to the scheme under consideration, should be mentioned.

Nonparametric estimators of the drift of a diffusion process have been studied by Dorogovtsev [1] and by Ibramkhalilov and Skorokhod [2]. Paper [3] deals with a nonparametric estimator of the intensity of an inhomogeneous Poisson process. A characteristic feature of the present paper is that the intensity function to be estimated enters nonlinearly into the drift coefficient of the observed process.

Denote $\mathbf{R}^+ := [0, \infty)$. Let $\{\Omega, \mathcal{F}, \mathcal{F}_t,\ t \geqq 0, \mathbf{P}\}$ be a stochastic basis with a probability measure $\mathbf{P}$ having no atom at infinity and let a trajectory of a stochastic process $\{\xi_t,\ t \geqq 0\}$, defined on this basis, be observed on an interval $[0, T]$. Suppose that the process has the representation

$$\xi_t = \xi_t(\lambda_0) := \xi_0 + \int_0^t a\bigl(s,\, \lambda_0(s)\bigr)\, ds + \int_0^t b(s)\, dw_s + N_t(\lambda_0), \tag{1}$$

where $a = a(s, x)\colon \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}$ and $b = b(s)\colon \mathbf{R}^+ \to \mathbf{R}$ are known nonrandom measurable functions, $\{w_t, \mathcal{F}_t,\ t \geqq 0\}$ is a Wiener process, and $\{N_t(\lambda_0), \mathcal{F}_t,\ t \geqq 0\}$ is an inhomogeneous Poisson process with unknown intensity $\lambda_0 = \lambda_0(t)$. We assume that the processes $\{w_t\}$, $\{N_t(\lambda_0)\}$ and the random variable $\xi_0$ are independent in totality. Suppose, in addition, that the nonrandom function $\lambda_0\colon \mathbf{R}^+ \to \mathbf{R}^+$, $\lambda_0 \in L^1_{\mathrm{loc}}$. The trajectories of $\{N_t\}$ belong with probability 1 to the space $X$ of step-functions $x$

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which are right-continuous and such that $x_0 = 0$, $\Delta x_t = 0$ or 1. The restriction of $X$ to the interval $[0,T]$ is denoted by $X_T$. Below, for brevity we write expressions of type $a(s,\lambda(s))$ under the integral sign as $a(s,\lambda)$.

From the probabilistic point of view the scheme of observations (1) is equivalent to the simultaneous observation over the two independent stochastic processes $\{N_t(\lambda_0), \mathcal{F}_t,\ t \geqq 0\}$ and $\{\eta_t(\lambda_0), \mathcal{F}_t,\ t \geqq 0\}$. Here

$$\eta_t(\lambda_0) := \xi_0 + \int_0^t a(s,\lambda_0)\,ds + \int_0^t b(s)\,dw_s;$$

let $Y_T := C[0,T]$ be the space to which the trajectories of the process on the interval $[0,T]$ belong. To the pair of processes $\{N_t(\lambda_0), \eta_t(\lambda_0),\ t \in [0,T]\}$ there corresponds a measure $\mu_T(\lambda_0)$ in the space $X_T \times Y_T$. Let $\lambda_1 \in L^1_{\mathrm{loc}}$ be another intensity of the Poisson component. According to [4, section 47, p. 264] and [5, section 8, Theorem 14, p. 343], given the conditions

$$\text{(a)} \qquad \int_0^T \frac{\lambda_1(s)}{\lambda_0(s)}\,ds < \infty,$$

$$\text{(b)} \qquad \int_0^T b^{-2}(s)\,\bigl|a(s,\,\lambda_1) - a(s,\,\lambda_0)\bigr|^2 ds < \infty,$$

the measure $\mu_T(\lambda_1) \ll \mu_T(\lambda_0)$, and the likelihood ratio has the form

$$\begin{aligned}
&\frac{d\mu_T(\lambda_1)}{d\mu_T(\lambda_0)}\,\bigl(N_T(\lambda_0),\ \eta_t(\lambda_0)\bigr) \\
&\quad = \exp\biggl\{\int_0^T b^{-1}(s)\,\bigl(a(s,\,\lambda_1) - a(s,\,\lambda_0)\bigr)\,dw_s \\
&\qquad\qquad - \frac{1}{2}\int_0^T b^{-2}(s)\,\bigl|a(s,\,\lambda_1) - a(s,\,\lambda_0)\bigr|^2 ds \\
&\qquad\qquad - \int_0^T \bigl(\lambda_1(s) - \lambda_0(s)\bigr)\,ds + \int_0^T \log\frac{\lambda_1(s)}{\lambda_0(s)}\,dN_s(\lambda_0)\biggr\}.
\end{aligned} \tag{2}$$

**2. Construction of the estimator.** Now the maximum likelihood estimator (MLE) of $\lambda_0$ is constructed by maximizing the right-hand side of (2). Introduce the needed spaces. For $r > 1$ denote by $\overline{W}{}^1_r[0,\tau]$ the Sobolev space of functions $\lambda$, specified on $[0,\tau]$, having the generalized derivative $\lambda' \in L_r[0,\tau]$ (see [6, p. 441] for the definition of generalized derivatives) and such that $\lambda(0) = \lambda(\tau)$:

$$\|\lambda\|_{\overline{W}{}^1_r} := \left(\int_0^\tau \bigl|\lambda(t)\bigr|^r dt + \int_0^\tau \bigl|\lambda'(t)\bigr|^r dt\right)^{1/r}.$$

*Remark* 1. The following Sobolev imbedding theorems are valid [6, pp. 439, 444]:

(a) For $r > 1$ the space $\overline{W}{}^1_r[0,\tau]$ is a compact imbedding into $C[0,\tau]$.

(b) For $1 < r < s$ the space $\overline{W}{}^1_s[0,\tau]$ is a compact imbedding into $\overline{W}{}^1_r[0,\tau]$.

We impose the following restrictions on the intensity function:

(c) $\lambda_0$ is periodic with the known period $\tau > 0$.

(d) The restriction of $\lambda_0$ to $[0,\tau]$ belongs to $K$, where $K$ is a compact in $\overline{W}{}^1_2[0,\tau]$, and all functions in $K$ are nonnegative.

We introduce several notations related to compact $K$:

$$\overline{K} := \sup_{\lambda \in K} \|\lambda\|_{\overline{W}_2^1}, \quad \lambda_*(s) := \min_{\lambda \in K} \lambda(s),\ \lambda^*(s) := \max_{\lambda \in K} \lambda(s).$$

In addition, let $\lambda_*$ and $\lambda^*$ be, respectively, the minimum of $\lambda_*(s)$ and the maximum of $\lambda^*(s)$ on $[0, \tau]$. Below, we make no difference between the functions from $K$ and their $\tau$-periodic continuations to $\mathbf{R}^+$. The functions $\lambda_*(s)$ and $\lambda^*(s)$ will be considered as continuous with period $\tau$ to $\mathbf{R}^+$.

Set

$$\text{(e)} \qquad \int_0^\tau \frac{ds}{\lambda_*(s)} < \infty.$$

Note that under conditions (c)–(e)

$$\mathbf{E} \int_0^\tau \big| \log \lambda_*(t) \big| \, dN_t(\lambda_0) = \int_0^\tau \big| \log \lambda_*(t) \big| \, \lambda_0(t) \, dt < \infty$$

and, therefore, for any function $\lambda_1 \in K$

$$\int_0^\tau \big| \log \lambda_1(t) \big| \, dN_t(\lambda_0) < \infty \quad \text{a.s.},$$

and the likelihood ratio (2) is well defined.

Set $\alpha = \alpha(s, x) := a(s, x)/b(s)$, $s \in \mathbf{R}^+$, $x \in \mathbf{R}^+$.

(f) For any function $\lambda \in K$, $\alpha(\cdot, \lambda(\cdot)) \in L^2_{\mathrm{loc}}$.

We construct the MLE of $\lambda_0$ by the observations of (1) on the interval $[0, T]$, $T \geqq \tau$. It follows from equality (2) that to this aim it suffices to consider the following functional, containing the integrals of the observed components $\{N_t(\lambda_0),\, \eta_t(\lambda_0)\}$:

$$\begin{aligned} Q_T(\lambda) = Q_T(\lambda,\, \omega) &:= \frac{1}{T} \int_0^T \alpha(s,\, \lambda) \, dw_s \\ &- \frac{1}{2T} \int_0^T \big[\alpha^2(s,\, \lambda) - 2\alpha(s,\, \lambda_0) \, \alpha(s,\, \lambda)\big] \, ds + \widetilde{Q}_T(\lambda). \end{aligned} \tag{3}$$

Here

$$\widetilde{Q}_T(\lambda) := -\frac{1}{T} \int_0^T \lambda(s) \, ds + \frac{1}{T} \int_0^T \log \lambda(s) \, dN_s(\lambda_0), \quad \lambda \in K.$$

If conditions (c)–(f) hold true, there exists a set $\Omega_0 \subset \Omega$ such that $\mathbf{P}(\Omega_0) = 1$ and $Q_T(\lambda, \omega)$ takes real values for any $T \geqq \tau$, $\omega \in \Omega_0$, $\lambda \in K$. The next assumption provides the continuity of the functional $Q_T$ in $\lambda$ for $\omega \in \Omega_0$.

(g) The function $\alpha \in C^1(\mathbf{R}^+ \times [\lambda_*,\, \lambda^*])$ and, for any $T > 0$, there exists $C_T > 0$ such that, for $s \leqq T$

$$\left| \frac{\partial \alpha(s, x)}{\partial s} - \frac{\partial \alpha(s, y)}{\partial s} \right| \leqq C_T |x - y|,$$
$$\left| \frac{\partial \alpha(s, x)}{\partial x} - \frac{\partial \alpha(s, y)}{\partial x} \right| \leqq C_T |x - y|.$$

Below, the notation $C_T$ or $D_T$ is used to denote a constant depending on $T$ only; the symbols $C$ or $C_1$, $C_2$ stand for absolute constants. One and the same letter may be used to denote different constants.

LEMMA 1. *Under conditions* (c)–(e) *and* (g) *the functional* $Q_T$ *is continuous in* $\lambda \in K$ *for* $\omega \in \Omega_0$.

*Proof.* The continuity of $\widetilde{Q}_T(\lambda)$ is evident. Further, condition (g) implies the boundedness of the functions $\alpha$ and $\partial\alpha/\partial s$ on the compact $\widetilde{K} := [0,T] \times [\lambda_*, \lambda^*]$. Hence for $\lambda_1, \lambda_2 \in K$

$$\left| \int_0^T \left[\alpha^2(s,\,\lambda_1) - \alpha^2(s,\,\lambda_2)\right] ds - 2\int_0^T \alpha(s,\,\lambda_0)\left[\alpha(s,\,\lambda_1) - \alpha(s,\,\lambda_2)\right] ds \right|$$

$$\leqq C_T \int_0^T \left|\lambda_1(s) - \lambda_2(s)\right| ds \leqq C_T\left(\left[\frac{T}{\tau}\right] + 1\right) \|\lambda_1 - \lambda_2\|_{\overline{W}_2^1}. \tag{4}$$

Here and in what follows $[T/\tau]$ is the integral part of the ratio $T/\tau$. The increment of the stochastic integral in (3) can be written like this:

$$\int_0^T \left[\alpha(s,\,\lambda_2) - \alpha(s,\,\lambda_1)\right] dw_s = w_T\Big[\alpha\big(T,\,\lambda_2(T)\big) - \alpha\big(T,\,\lambda_1(T)\big)\Big]$$

$$- \int_0^T w_s \left[\frac{\partial\alpha(s,\lambda_2)}{\partial s} - \frac{\partial\alpha(s,\lambda_1)}{\partial s} + \frac{\partial\alpha(s,\lambda_2)}{\partial x}\,\lambda_2'(s) - \frac{\partial\alpha(s,\lambda_1)}{\partial x}\,\lambda_1'(s)\right] ds. \tag{5}$$

Hence, by using Remark 1(a) we have

$$\left|\int_0^T \left[\alpha(s,\,\lambda_2) - \alpha(s,\,\lambda_1)\right] dw_s\right| \leqq |w_T|\, C_T\left|\lambda_2(T) - \lambda_1(T)\right|$$

$$+ C_T \int_0^T |w_s|\left|\lambda_2(s) - \lambda_1(s)\right| ds + \int_0^T |w_s| \left|\frac{\partial\alpha(s,\lambda_2)}{\partial x} - \frac{\partial\alpha(s,\lambda_1)}{\partial x}\right| \left|\lambda_2'(s)\right| ds$$

$$+ C_T \int_0^T |w_s|\left|\lambda_2'(s) - \lambda_1'(s)\right| ds \leqq \widetilde{C}_T\Big(|w_T| + \overline{K}\sup_{s\leqq T}|w_s|\Big)\, \|\lambda_2 - \lambda_1\|_{\overline{W}_2^1}, \tag{6}$$

where $\widetilde{C}_T$ is a constant being, generally speaking, other than $C_T$. Recall that $\overline{K}$ was defined just after condition (d). Estimates (4)–(6) imply the desired statement.

The proof of the next result is based on applying the measurable choice theorem [7] and is similar to the proof of existence of an estimator in [8].

LEMMA 2. *Let conditions* (c)–(e) *and* (g) *hold. Then for each* $T \geqq \tau$ *there exists a random element* $\lambda_T = \lambda_T(\omega, s)$, $\omega \in \Omega$, $s \in [0,T]$, *belonging to* $K$, *such that the relation*

$$Q_T(\lambda_T) = \max_{\lambda\in K} Q_T(\lambda)$$

*is satisfied for each* $\omega \in \Omega_0$.

DEFINITION. *The random function* $\lambda_T$ *from Lemma* 2 (*extended to* $\mathbf{R}^+$ *by period* $\tau$) *is called* a maximum likelihood estimator of the intensity of the process $\{N_t(\lambda_0)\}$ constructed by the observations $\{\xi_t,\ t \in [0,T]\}$.

*Remark* 2. Generally speaking, $\lambda_T$ is not specified uniquely. Below, we take an arbitrary function satisfying the conditions above as the MLE.

*Remark* 3. An algorithm of approximate calculation of a nonparametric MLE constructed by observations over the Poisson component was suggested in [3]. A similar procedure can be applied to the introduced estimators $\lambda_T$ in the combined model of observations.

*Remark* 4. Since $\lambda_T(s)$ is a looking-ahead function, the summand

$$\int_0^T \alpha(s, \lambda_T(s))\, dw_s,$$

entering $Q_T(\lambda_T)$, is not an Itô stochastic integral. However, under restriction (g) on the smoothness of $\alpha$ and the appropriate smoothness of $\lambda_T(s)$ this integral can be treated by means of integration by parts similarly to (5).

**3. Consistency of the estimator.** Assume that $\alpha$ meets the following conditions:

$$\text{(h)} \qquad \alpha(s,\, x) = \sum_{i=1}^{3} \alpha_i(s,\, x), \quad \text{where } \alpha_i \in C\big(\mathbf{R}^+ \times [\lambda_*, \lambda^*]\big)$$

and, in addition,

(h1) for each $x \in [\lambda_*, \lambda^*]$ $\alpha_1$ is a $\tau$-periodic in $s$;

$$\text{(h2)} \qquad \max_{\lambda_*(s) \leqq x \leqq \lambda^*(s)} |\alpha_2(s, x)| \longrightarrow 0, \quad s \to \infty;$$

$$\text{(h3)} \qquad \int_1^\infty \frac{(\alpha_3^*(s))^2}{s}\, ds < \infty, \quad \text{where } \alpha_3^*(s) := \max_{\lambda_*(s) \leqq x \leqq \lambda^*(s)} \big|\alpha_3(s,\, x)\big|.$$

Thus, $\alpha$ is decomposed into three components: periodic in $s$, uniformly vanishing as $s \to \infty$, and small “in average” as $s \to \infty$. Introduce the limiting functional

$$Q_\infty(\lambda) := -\frac{1}{2\tau} \int_0^\tau \big|\alpha_1(s,\, \lambda)\big|^2 ds + \frac{1}{\tau} \int_0^\tau \alpha_1(s,\, \lambda)\, \alpha_1(s,\, \lambda_0)\, ds + \widetilde{Q}_\infty(\lambda),$$

where

$$\widetilde{Q}_\infty(\lambda) := -\frac{1}{\tau} \int_0^\tau \lambda(s)\, ds + \frac{1}{\tau} \int_0^\tau \log \lambda(s)\, \lambda_0(s)\, ds; \quad \lambda \in K.$$

THEOREM 1. *Let conditions* (c)–(e), (g), *and* (h) *be fulfilled. Then for each* $\lambda \in K$

$$Q_T(\lambda) \to Q_\infty(\lambda) \quad (a.s.), \quad T \to \infty.$$

*Proof.* As shown in [3], conditions (c)–(e) imply $\widetilde{Q}_T(\lambda) \to \widetilde{Q}_\infty(\lambda)$ a.s. as $T \to \infty$. Consider the convergence of the remaining summands of functional (3). Set

$$Q_T^i(\lambda) := \frac{1}{T} \int_0^T \alpha_i(s,\, \lambda)\, dw_s, \quad \lambda \in K,\ 1 \leqq i \leqq 3.$$

For $T = n\tau$ and $i = 1$

$$Q_{n\tau}^1(\lambda) = \frac{1}{n\tau} \sum_{i=1}^{n} \int_{(i-1)\tau}^{i\tau} \alpha_1(s,\, \lambda)\, dw_s \longrightarrow 0 \quad \text{(a.s.)}, \quad n \to \infty,$$

by the strong law of large numbers (SLLN) [9, Theorem 12, p. 332] and the inequality $\mathbf{E}|\int_0^\tau \alpha_1(s, \lambda)\, dw_s| \leqq (\int_0^\tau \alpha_1^2(s, \lambda)\, ds)^{1/2} < \infty$.

To estimate "the remainder" $|Q^1_T(\lambda) - Q^1_{n\tau}(\lambda)|$ it suffices to evaluate the expression $n^{-1}\sup_{0\leqq h\leqq\tau}|\int_{n\tau}^{n\tau+h}\alpha_1(s,\lambda)\,dw_s|$. By the martingale Doob inequality [10, p. 67] and the periodicity of $\alpha_1$ we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\mathbf{E}\sup_{0\leqq h\leqq\tau}\left|\int_{n\tau}^{n\tau+h}\alpha_1(s,\,\lambda)\,dw_s\right|^2\leqq 4\sum_{n=1}^{\infty}\frac{1}{n^2}\int_0^{\tau}\alpha_1^2(s,\,\lambda)\,ds<\infty.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sup_{0\leqq h\leqq\tau}\left|\int_{n\tau}^{n\tau+h}\alpha_1(s,\,\lambda)\,dw_s\right|^2<\infty \quad \text{(a.s.)},$$

whence

$$\frac{1}{n}\sup_{0\leqq h\leqq\tau}\left|\int_{n\tau}^{n\tau+h}\alpha_1(s,\,\lambda)\,dw_s\right|\longrightarrow 0 \quad \text{(a.s.)},\ n\to\infty. \tag{7}$$

Thus, $Q^1_T(\lambda)\to 0$ (a.s.), $T\to\infty$.

Further, by condition (h2)

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\mathbf{E}\left|\int_{(n-1)\tau}^{n\tau}\alpha_2(s,\,\lambda)\,dw_s\right|^2=\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{(n-1)\tau}^{n\tau}\left|\alpha_2(s,\,\lambda)\right|^2ds<\infty.$$

Applying the SLLN [9, p. 332, Theorem 12] once again we obtain that $Q^2_{n\tau}(\lambda)\to 0$ a.s. as $n\to\infty$. "The remainder" $|Q^2_T(\lambda)-Q^2_{n\tau}(\lambda)|$ is estimated similarly to (7); as a result we see that $Q^2_T(\lambda)\to 0$ a.s. as $T\to\infty$.

Further, by condition (h3) we have

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\int_{(n-1)\tau}^{n\tau}\alpha_3^2(s,\,\lambda)\,ds\leqq\tau^2\int_{\tau}^{\infty}\frac{\alpha_3^2(s,\,\lambda)}{s^2}\,ds+\int_0^{\tau}\alpha_3^2(s,\,\lambda)\,ds<\infty$$

and, according to the SLLN [9, Theorem 12, p. 332], $Q^3_{n\tau}(\lambda)\to 0$ a.s. as $n\to\infty$. "The remainder" $|Q^3_T(\lambda)-Q^3_{n\tau}(\lambda)|$ is estimated similarly to (7), and we conclude that $Q^3_T(\lambda)\to 0$ a.s. as $T\to\infty$.

Finally, consider

$$R_T(\lambda):=\frac{1}{T}\int_0^T\left[\alpha^2(s,\,\lambda)-2\alpha(s,\,\lambda)\,\alpha(s,\,\lambda_0)\right]ds. \tag{8}$$

Under condition (h) we have as $T\to\infty$

$$\frac{1}{T}\int_0^T\alpha_1^2(s,\,\lambda)\,ds\longrightarrow\frac{1}{\tau}\int_0^{\tau}\alpha_1^2(s,\,\lambda)\,ds;\qquad \frac{1}{T}\int_0^T\alpha_2^2(s,\,\lambda)\,ds\longrightarrow 0;$$

$$\begin{aligned}\lim_{T\to\infty}\frac{1}{T}\int_1^T\alpha_3^2(s,\,\lambda)\,ds &= \lim_{T\to\infty}\left[\int_1^T\frac{\alpha_3^2(s,\,\lambda)}{s}\,ds-\frac{1}{T}\int_1^T\left(\int_1^z\frac{\alpha_3^2(s,\,\lambda)}{s}\,ds\right)dz\right]\\ &= \int_0^{\infty}\frac{\alpha_3^2(s,\,\lambda)}{s}\,ds-\lim_{z\to\infty}\int_1^z\frac{\alpha_3^2(s,\,\lambda)}{s}\,ds=0.\end{aligned}$$

The remaining terms in (8) are estimated in a similar way and we have as a result that as $T\to\infty$

$$R_T(\lambda)\longrightarrow\frac{1}{\tau}\int_0^{\tau}\left|\alpha_1(s,\,\lambda)\right|^2ds-\frac{2}{\tau}\int_0^{\tau}\alpha_1(s,\,\lambda)\,\alpha_1(s,\,\lambda_0)\,ds.$$

Now convergence of $Q_T(\lambda)$ to $Q_\infty(\lambda)$ follows from the obtained limiting relationships.

THEOREM 2. *Let conditions* (c)–(e), (g), *and* (h), *hold and, in addition, condition* (i):

(i1) $\alpha_1 \in C^1([0,\tau] \times [\lambda_*, \lambda^*])$;

(i2) $\alpha_i \in C^1(\mathbf{R}^+ \times [\lambda_*, \lambda^*])$ *and for a bounded function* $C_s$

$$\left|\frac{\partial \alpha_i(s,x)}{\partial x}\right| \leqq C_s,$$

$$\left|\frac{\partial \alpha_i(s,x)}{\partial p} - \frac{\partial \alpha_i(s,y)}{\partial p}\right| \leqq C_s|x-y|,$$

*where* $p = s, x$; $i = 2, 3$; $s \in \mathbf{R}^+$; $x, y \in [\lambda_*, \lambda^*]$; *moreover,*

(i3) $$\limsup_{T\to\infty} \frac{1}{T} \int_e^T s(\log\log s)\, C_s^2 ds < \infty \quad \textit{and}$$

(i4) $$\lim_{t\to\infty} \sqrt{\frac{\log\log t}{t}}\, \alpha_3^*(t) = 0.$$

*Then the estimator* $\lambda_T$ *is consistent in the following sense*:

(9) $$\|\lambda_T - \lambda_0\|_{\overline{W}_2^1} \longrightarrow 0 \quad a.s., \ T \to \infty.$$

*Proof.* According to the general conditions of consistency [1, p. 76], in order that (9) be true it suffices that the following conditions are satisfied: (a) $Q_T(\lambda) \to Q_\infty(\lambda)$, $T \to \infty$ a.s.

(10) $$\text{(b)} \qquad Q_\infty(\lambda_0) > Q_\infty(\lambda), \quad \lambda \in K, \ \lambda \neq \lambda_0;$$

(c) there exists a function $\omega(\gamma)$, $\gamma > 0$, such that $\omega(\gamma) \to 0$, $\gamma \to 0$, and

(11) $$\mathbf{P}\left\{\limsup_{T\to\infty} \sup_{\lambda_1,\lambda_2\in K, \|\lambda_1-\lambda_2\|_{\overline{W}_2^1} \leqq \gamma} \left|Q_T(\lambda_2) - Q_T(\lambda_1)\right| \leqq \omega(\gamma)\right\} = 1.$$

Convergence (a) takes place by Theorem 1; inequality (10) follows from the representation

$$Q_\infty(\lambda) = -\frac{1}{2\tau}\int_0^\tau \left|\alpha_1(s,\,\lambda) - \alpha_1(s,\,\lambda_0)\right|^2 ds + \widetilde{Q}_\infty(\lambda) + \frac{1}{2\tau}\int_0^\tau \alpha_1^2(s,\,\lambda_0)\, ds$$

and the strict inequality $\widetilde{Q}_\infty(\lambda) < \widetilde{Q}_\infty(\lambda_0)$, $\lambda \neq \lambda_0$.

Relationship (11) for the functional $\widetilde{Q}_T(\lambda)$ has been proved in [3]. Thus, it remains to demonstrate the validity of (11) for the functional

$$\check{Q}_T(\lambda) := \frac{1}{T}\int_0^T \left|\alpha(s,\,\lambda) - \alpha(s,\,\lambda_0)\right|^2 ds$$

and the functionals $Q_T^i$ introduced while proving Theorem 1.

1) we evaluate $Q^1_T(\lambda_1) - Q^1_T(\lambda_2)$ for $T = n\tau$:

$$\begin{aligned} \left|Q^1_{n\tau}(\lambda_1) - Q^1_{n\tau}(\lambda_2)\right| &\leqq \frac{1}{n\tau}\left|B_n(\tau)\right| \left|\alpha_1\bigl(\tau,\, \lambda_1(\tau)\bigr) - \alpha_1\bigl(\tau,\, \lambda_2(\tau)\bigr)\right| \\ &+ \frac{1}{n\tau}\int_0^\tau \left|B_n(s)\right| \left|\frac{d}{ds}\Bigl(\alpha_1\bigl(s,\, \lambda_1(s)\bigr) - \alpha_1\bigl(s,\, \lambda_2(s)\bigr)\Bigr)\right| ds, \end{aligned} \tag{12}$$

where

$$B_n(s) := \sum_{i=0}^{n-1}\bigl(w(s+i\tau) - w(i\tau)\bigr), \quad s \in [0,\, \tau]. \tag{13}$$

By the SLLN [9, Theorem 12, p. 332] $n^{-1}|B_n(\tau)| \to 0$ a.s., $n \to \infty$, and the first summand in the right-hand side of (12) tends to zero a.s. uniformly in $\lambda_1, \lambda_2 \in K$. Further,

$$\begin{aligned} &\frac{1}{n\tau}\int_0^\tau \left|B_n(s)\right| \left|\frac{d}{ds}\Bigl(\alpha_1\bigl(s,\, \lambda_1(s)\bigr) - \alpha_1\bigl(s,\, \lambda_2(s)\bigr)\Bigr)\right| ds \\ &\leqq \frac{1}{n\tau}\left(\int_0^\tau B_n^2(s)\, ds\right)^{1/2} \left(\int_0^\tau \left|\frac{d}{ds}\,\Bigl(\alpha_1(s,\, \lambda_1(s)) - \alpha_1\bigl(s,\, \lambda_2(s)\bigr)\Bigr)\right|^2 ds\right)^{1/2}. \end{aligned} \tag{14}$$

Denote by $Y_i(s) := w(s+i\tau) - w(i\tau)$, $s \in [0, \tau]$, independent centered identically distributed Gaussian elements in the space $L_2[0,\tau]$. According to the SLLN for Hilbert spaces established in [11, p. 145] we have

$$\frac{1}{n}\left(\int_0^\tau B_n^2(s)\, ds\right)^{1/2} = \left\|\frac{1}{n}\sum_{i=0}^{n-1} Y_i\right\|_{L_2[0,\tau]} \longrightarrow 0 \quad \text{a.s.}, \quad n \to \infty.$$

Moreover, by condition (i1)

$$\begin{aligned} \int_0^\tau \left|\frac{d}{ds}\,\alpha_1\bigl(s,\, \lambda_1(s)\bigr)\right|^2 ds &\leqq 2\int_0^\tau \left|\frac{\partial\alpha_1(s,\lambda_1)}{\partial s}\right|^2 ds + 2\int_0^\tau \left|\frac{\partial\alpha_1(s,\lambda_1)}{\partial x}\right|^2 \left|\lambda_1'(s)\right|^2 ds \\ &\leqq C_1 + C_2\|\lambda_1\|^2_{W_2^1} \leqq C. \end{aligned} \tag{15}$$

A similar estimate is valid for the integral

$$\int_0^\tau \left|\frac{d}{ds}\,\alpha_1\bigl(s,\, \lambda_2(s)\bigr)\right|^2 ds.$$

Thus, estimates (12)–(15) imply

$$\limsup_{n\to\infty}\ \sup_{\lambda_1,\lambda_2\in K} \left|Q^1_{n\tau}(\lambda_1) - Q^1_{n\tau}(\lambda_2)\right| = 0 \quad \text{a.s.}$$

On account of (15) we evaluate “the remainder” as follows:

$$\begin{aligned} \sup_{\lambda\in K}\ \sup_{0\leqq h\leqq\tau} \frac{1}{n}\left|\int_{n\tau}^{n\tau+h} \alpha_1(s,\, \lambda)\, dw_s\right| &\leqq C \sup_{0\leqq h\leqq\tau} \frac{1}{n}\Bigl(\left|w(n\tau+h)\right| + \left|w(n\tau)\right|\Bigr) \\ &+ C\,\frac{1}{n}\left(\int_{n\tau}^{(n+1)\tau} w_s^2 ds\right)^{1/2}. \end{aligned} \tag{16}$$

According to the law of the iterated logarithm

$$\mathbf{P}\left\{ \limsup_{t\to\infty} \frac{|w(t)|}{\sqrt{t\log\log t}} < \infty \right\} = 1.$$

Estimating $|w(t)|$ from above by $\sqrt{t\log\log t}$, it is not difficult to show that the right-hand side of (16) vanishes a.s. as $n \to \infty$. Thus, we have proved that

$$\limsup_{T\to\infty} \sup_{\lambda_1,\lambda_2\in K} \left|Q_T^1(\lambda_1) - Q_T^1(\lambda_2)\right| = 0 \quad \text{a.s.}$$

2) Now we evaluate $|Q_T^2(\lambda_1) - Q_T^2(\lambda_2)|$:

$$\begin{aligned} \left|Q_T^2(\lambda_1) - Q_T^2(\lambda_2)\right| &\leqq \frac{1}{T}\,|w_T| \Big|\alpha_2\big(T,\,\lambda_1(T)\big) - \alpha_2\big(T,\,\lambda_2(T)\big)\Big| \\ &\quad + \frac{1}{T}\int_0^T |w_s| \left|\frac{d}{ds}\big(\alpha_2(s,\,\lambda_1) - \alpha_2(s,\,\lambda_2)\big)\right| ds. \end{aligned} \tag{17}$$

By condition (h2) the first summand in the right-hand side of (17) tends to zero a.s. as $T \to \infty$ uniformly in $\lambda_1, \lambda_2 \in K$. The second summand does not exceed the sum

$$\begin{aligned} &\frac{1}{T}\int_0^T |w_s| \left|\frac{\partial}{\partial s}\alpha_2(s,\,\lambda_1) - \frac{\partial}{\partial s}\alpha_2(s,\,\lambda_2)\right| ds \\ &\quad + \frac{1}{T}\int_0^T |w_s| \left|\frac{\partial}{\partial x}\alpha_2(s,\,\lambda_1) - \frac{\partial}{\partial x}\alpha_2(s,\,\lambda_2)\right| \left|\lambda_1'(s)\right| ds \\ &\quad + \frac{1}{T}\int_0^T |w_s| \left|\frac{\partial}{\partial x}\alpha_2(s,\,\lambda_2)\right| \left|\lambda_2'(s) - \lambda_1'(s)\right| ds. \end{aligned} \tag{18}$$

In view of condition (i2) we have for $T \in [n\tau, (n+1)\,\tau]$

$$\frac{1}{T}\int_0^T |w_s| \left|\frac{\partial}{\partial s}\alpha_2(s,\,\lambda_1) - \frac{\partial}{\partial s}\alpha_2(s,\,\lambda_2)\right| ds \leqq \frac{C}{\sqrt{T}} \left(\int_0^T w_s^2 C_s^2 ds\right)^{1/2} \|\lambda_2 - \lambda_1\|_{\overline{W}_2^1}.$$

Note that $T^{-1}\int_0^A w_s^2 ds \longrightarrow 0$ a.s., $T \to \infty$, for each $A > 0$ and by virtue of conditions (i2) and (i3) and the law of the iterated logarithm with probability 1

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T w_s^2 C_s^2 ds \leqq \limsup_{T\to\infty} \frac{2}{T}\int_e^T s\,(\log\log s)\,C_s^2 ds < \infty. \tag{19}$$

Furthermore, according to Remark 1(a)

$$\begin{aligned} &\frac{1}{T}\int_0^T |w_s| \left|\frac{\partial\alpha_2(s,\,\lambda_1)}{\partial x} - \frac{\partial\alpha_2(s,\,\lambda_2)}{\partial x}\right| \left|\lambda_1'(s)\right| ds \\ &\quad \leqq \frac{1}{T}\int_0^T |w_s|\, C_s \left|\lambda_1(s) - \lambda_2(s)\right| \left|\lambda_1'(s)\right| ds \\ &\quad \leqq C\,\overline{K}\left(\frac{1}{T}\int_0^T w_s^2 C_s^2 ds\right)^{1/2} \|\lambda_2 - \lambda_1\|_{\overline{W}_2^1}, \end{aligned}$$

which combined with (19) gives the needed estimates. Here $\overline{K}$ is the constant defined in item 2).

Finally,

$$\begin{aligned}
&\frac{1}{T}\int_0^T |w_s|\left|\frac{\partial\alpha_2(s,\,\lambda_2)}{\partial x}\right|\left|\lambda_2'(s)-\lambda_1'(s)\right|ds\\
&\quad\leqq C\left(\frac{1}{T}\int_0^T w_s^2\left|\frac{\partial\alpha_2(s,\,\lambda_2)}{\partial x}\right|^2 ds\right)^{1/2}\|\lambda_2-\lambda_1\|_{\overline{W}_2^1}\\
&\quad\leqq C\left(\frac{1}{T}\int_0^T w_s^2\,C_s^2 ds\right)^{1/2}\|\lambda_2-\lambda_1\|_{\overline{W}_2^1}.
\end{aligned}$$

The estimates obtained yield relationship (11) for $Q_T^2(\lambda)$ with linear function $\omega(\gamma)$.

Similarly, by using conditions (i2)–(i4) one can estimate the increments of the component $Q_T^3(\lambda)$.

3) Let us evaluate

$$\begin{aligned}
&\left|\check{Q}_T(\lambda_1)-\check{Q}_T(\lambda_2)\right|\\
&\quad\leqq 4\left(\frac{1}{T}\int_0^T\left|\alpha(s,\,\lambda_1)-\alpha(s,\,\lambda_2)\right|^2 ds\right)^{1/2}\sup_{\lambda\in K}\left(\frac{1}{T}\int_0^T\left|\alpha(s,\,\lambda)\right|^2 ds\right)^{1/2}.
\end{aligned}$$

Under the conditions of the theorem there exists $C>0$ such that

$$\frac{1}{T}\int_0^T\left|\alpha_i(s,\,\lambda)\right|^2 ds\leqq C,\quad 1\leqq i\leqq 3,$$

$$\frac{1}{T}\int_0^T\left|\alpha(s,\,\lambda_1)-\alpha(s,\lambda_2)\right|^2 ds\leqq C\,\|\lambda_2-\lambda_1\|_{\overline{W}_2^1}^2,$$

whence we obtain (11) for $\check{Q}_T(\lambda)$ and the theorem is proved.

*Remark* 5. Let $K$ be a compact in $\overline{W}_{2p}^1$ for some $p>1$, and all the conditions of Theorem 2 hold. Then estimator $\lambda_T$ is consistent in $\overline{W}_{2p}^1$, that is, $\|\lambda_T-\lambda_0\|_{\overline{W}_{2p}^1}\longrightarrow 0$ a.s., $T\to\infty$. Indeed, by virtue of Remark 1(a)

$$\|f\|_{\overline{W}_2^1}\leqq C_p\|f\|_{\overline{W}_{2p}^1},\quad f\in\overline{W}_2^1;$$

consequently,

$$\begin{aligned}
&\mathbf{P}\left\{\limsup_{T\to\infty}\sup_{\lambda_1,\lambda_2\in K,\,\|\lambda_1-\lambda_2\|_{\overline{W}_{2p}^1}\leqq\gamma}\left|Q_T(\lambda_2)-Q_T(\lambda_1)\right|\leqq\omega(\gamma)\right\}\\
&\quad\geqq\mathbf{P}\left\{\limsup_{T\to\infty}\sup_{\lambda_1,\lambda_2\in K,\,\|\lambda_1-\lambda_2\|_{\overline{W}_2^1}\leqq C_p\gamma}\left|Q_T(\lambda_2)-Q_T(\lambda_1)\right|\leqq\omega(\gamma)\right\}=1.
\end{aligned}$$

In addition, $Q_T$ is a continuous functional in $\lambda\in\overline{W}_{2p}^1$; now consistency in $\overline{W}_{2p}^1$ follows from [1, p. 76].

**4. The rate of the mean square convergence of the estimator.** Introduce additional restrictions on the parametric set $K$ and the intensity function $\lambda_0$.

(j) For some $p\in(1,2)$, $K$ is a compact set in $\overline{W}_{2p}^1[0,\tau]$ and is bounded in $\overline{W}_{2q}^1[0,\tau]$, where $p,q$ are conjugate indices; moreover, all functions in $K$ are nonnegative.

(k) The function $\lambda_0$ belongs to $K$ and is positive for each $t\in[0,\tau]$.

In what follows we denote by $O_p(1)$ a stochastic process $\{x_T,\ T \geqq 0\}$ if

$$\lim_{C\to\infty} \sup_{T\geqq T_0} \mathbf{P}\{|x_T| \geqq C\} = 0 \quad \text{for some } T_0 > 0,$$

and by $o_p(1)$ if $x_T \xrightarrow{\mathbf{P}} 0$, $T \to \infty$. While using the introduced symbols we omit the indication that $T \to \infty$.

THEOREM 3. *Let conditions* (c), (e), (g)–(i), (j), *and* (k) *be fulfilled and, in addition, the following conditions* (l) *hold.*

(l1) *There exists a constant $C$ such that*

$$\left|\frac{\partial \alpha_1(s,x)}{\partial p} - \frac{\partial \alpha_1(s,y)}{\partial p}\right| \leqq C\,|x-y|,$$

*where $p = s, x$; $s \in [0,\tau]$; $x, y \in [\lambda_*, \lambda^*]$;*

(l2) $\int_0^\infty s\,C_s^2 ds < \infty$, *where $C_s$ is taken from condition* (i) *and, in addition, the inequalities*

$$\left|\frac{\partial \alpha_i(s,x)}{\partial s}\right| \leqq C_s, \quad i = 2,\,3,\ s \in \mathbf{R}^+,\ x \in [\lambda_*,\,\lambda^*],$$

*are valid. Then*

$$T^{1/2}\int_0^\tau \bigl|\lambda_T(t) - \lambda_0(t)\bigr|^2 dt = o_p(1).$$

*Proof.* According to Theorem 2 and Remark 5, given condition (j) $\|\lambda_T - \lambda_0\|_{\overline{W}^1_{2p}} \longrightarrow 0$ a.s. as $T \to \infty$. By the imbedding theorem for Sobolev spaces (see Remark 1(b)) and condition (k) we have

$$\max_{0\leqq t\leqq\tau} \frac{|\lambda_T(t) - \lambda_0(t)|}{\lambda_0(t)} \longrightarrow 0 \quad \text{(a.s.)},\ T \to \infty. \tag{20}$$

Denote

$$\Delta_T(t) := \lambda_T(t) - \lambda_0(t); \qquad a_0(t) := \int_0^t \lambda_0(u)\,du;$$
$$N_i(t) := N\bigl(t + \tau(i-1)\bigr) - N\bigl(\tau(i-1)\bigr), \quad i \geqq 1,$$

where $N(t) = N_t(\lambda_0)$; $A_n(t) := n^{-1}\sum_{i=1}^n N_i(t) - a_0(t)$; $t \in [0,\tau]$. We rewrite the functional $Q_T$ in the form

$$\begin{aligned} Q_T(\lambda) = {}& \frac{1}{T}\int_0^{n\tau} \alpha(s,\,\lambda)\,dw_s - \frac{1}{2T}\int_0^T \bigl|\alpha(s,\,\lambda) - \alpha(s,\,\lambda_0)\bigr|^2 ds + \frac{1}{2T}\int_0^T \alpha^2(s,\,\lambda_0)\,ds \\ & + \frac{n}{T}\biggl[-\int_0^\tau \lambda(s)\,ds + \int_0^\tau \lambda_0(t)\,\log\lambda(t)\,dt + \int_0^\tau \log\lambda(t)\,dA_n(t)\biggr] \\ & + R_T^1(\lambda) + R_T^2(\lambda), \end{aligned}$$

where

$$R_T^1(\lambda) := -\frac{1}{T}\int_0^{T-n\tau} \lambda(t)\,dt + \frac{1}{T}\int_0^{T-n\tau} \log\lambda(t)\,dN_{n+1}(t),$$
$$R_T^2(\lambda) := \frac{1}{T}\int_{n\tau}^T \alpha(s,\,\lambda)\,dw_s, \quad n\tau \leqq T < (n+1)\,\tau.$$

From the definition of the estimator we see that $Q_T(\lambda_T) \geqq Q_T(\lambda_0)$, whence

$$\begin{aligned}\int_0^\tau \Delta_T(t)\,dt - \int_0^\tau \lambda_0(t)\,\log\left(1+\frac{\Delta_T(t)}{\lambda_0(t)}\right)dt \\ \leqq \int_0^\tau \log\left(1+\frac{\Delta_T(t)}{\lambda_0(t)}\right)dA_n(t) + \frac{1}{n}\int_0^{n\tau}\left[\alpha(s,\,\lambda_T)-\alpha(s,\,\lambda_0)\right]dw_s \\ -\frac{1}{2n}\int_0^{n\tau}\left|\alpha(s,\,\lambda_T)-\alpha(s,\,\lambda_0)\right|^2 ds + \frac{T}{n}\sum_{i=1,\,2}\Big(\left|R_T^i(\lambda_T)\right| + \left|R_T^i(\lambda_0)\right|\Big).\end{aligned}$$

As shown in [3], under the conditions of the theorem $R_T^1(\lambda_T)$ and $R_T^1(\lambda_0)$ are $O_p(1)/T$. Moreover, it follows from (20) that there exists a $\delta > 0$ such that

$$\lim_{T\to\infty}\mathbf{P}\left\{\int_0^\tau \left[\Delta_T(t) - \lambda_0(t)\,\log\left(1+\frac{\Delta_T(t)}{\lambda_0(t)}\right)\right]dt \geqq \delta\int_0^\tau \left|\Delta_T(t)\right|^2 dt\right\} = 1. \tag{21}$$

In [3] it is also established that

$$\int_0^\tau \log\left(1+\frac{\Delta_T(t)}{\lambda_0(t)}\right)dA_n(t) = \frac{o_p(1)}{\sqrt{T}}. \tag{22}$$

We evaluate

$$Q_{n\tau}^i(\lambda_T) - Q_{n\tau}^i(\lambda_0) = \frac{1}{n\tau}\int_0^{n\tau}\left[\alpha_i(s,\,\lambda_T)-\alpha_i(s,\,\lambda_0)\right]dw_s,$$

using (12)–(15), (17), (18), and conditions (l):

$$\begin{aligned}\left|Q_{n\tau}^1(\lambda_T) - Q_{n\tau}^1(\lambda_0)\right| \leqq \frac{1}{n\tau}\left|B_n(\tau)\right|C\left|\lambda_T(\tau)-\lambda_0(\tau)\right| + \frac{1}{\sqrt{n}\,\tau}\left\|\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}Y_i\right\|_{L_2[0,\tau]} \\ \times\Bigg(\int_0^\tau\Bigg[\left|\frac{\partial}{\partial s}\,\alpha_1(s,\,\lambda_T) - \frac{\partial}{\partial s}\,\alpha_1(s,\,\lambda_0)\right| \\ + \left|\frac{\partial}{\partial x}\,\alpha_1(s,\,\lambda_T) - \frac{\partial}{\partial x}\,\alpha_1(s,\,\lambda_0)\right|\left|\lambda_0'(s)\right| \\ + \left|\frac{\partial}{\partial x}\,\alpha_1(s,\,\lambda_T)\right|\left|\lambda_T'(s)-\lambda_0'(s)\right|\Bigg]^2 ds\Bigg)^{1/2}.\end{aligned} \tag{23}$$

We know that

$$\frac{1}{\sqrt{n}}\left|B_n(\tau)\right| = O_p(1), \qquad \left\|\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}Y_i\right\|_{L_2[0,\tau]} = O_p(1). \tag{24}$$

By condition (l1)

$$\left(\int_0^\tau \left|\frac{\partial}{\partial s}\big(\alpha_1(s,\,\lambda_T)-\alpha_1(s,\,\lambda_0)\big)\right|^2 ds\right)^{1/2} = o_p(1). \tag{25}$$

According to conditions (j) and (l1)

$$\begin{aligned}&\left(\int_0^\tau \left|\frac{\partial}{\partial x}\big(\alpha_1(s,\,\lambda_T)-\alpha_1(s,\,\lambda_0)\big)\right|^2 \big|\lambda_0'(s)\big|^2 ds\right)^{1/2}\\&\quad\leqq C\left(\int_0^\tau |\lambda_T-\lambda_0|^{2p} ds\right)^{1/(2p)}\left(\int_0^\tau |\lambda_0'(s)|^{2q} ds\right)^{1/(2q)}\\&\quad\leqq C\,\|\lambda_T-\lambda_0\|_{\overline{W}^1_{2p}}=o_p(1).\end{aligned}\tag{26}$$

Additionally,

$$\left(\int_0^\tau \left|\frac{\partial}{\partial x}\,\alpha_1(s,\,\lambda_T)\right|^2 \big|\lambda_T'(s)-\lambda_0'(s)\big|^2 ds\right)^{1/2}\leqq C\,\|\lambda_T-\lambda_0\|_{\overline{W}^1_{2p}}=o_p(1).\tag{27}$$

In view of (24)–(27) the left-hand side of (23) is $o_p(1)/\sqrt{T}$. Relations (17), (18), and condition (l2) give for $i=2,3$

$$\begin{aligned}\big|Q^i_{n\tau}(\lambda_T)-Q^i_{n\tau}(\lambda_0)\big| &\leqq \frac{1}{\sqrt{n\tau}}\,\frac{|w_{n\tau}|}{\sqrt{n\tau}}\,\Big|\alpha_i\big(n\tau,\,\lambda_T(n\tau)\big)-\alpha_i\big(n\tau,\,\lambda_0(n\tau)\big)\Big|\\&\quad+\frac{C}{\sqrt{n}}\left(\int_0^{n\tau} w_s^2C_s^2 ds\right)^{1/2}\|\lambda_T-\lambda_0\|_{\overline{W}^1_2}\\&\quad+\frac{C}{n}\left(\int_0^{n\tau} w_s^2C_s^2 ds\right)^{1/2}\|\lambda_T-\lambda_0\|_{\overline{W}^1_2}=\frac{o_p(1)}{\sqrt{T}}.\end{aligned}\tag{28}$$

We make use of the estimate

$$\mathbf{E}\int_0^{n\tau} w_s^2C_s^2 ds=\int_0^{n\tau} s\,C_s^2 ds\leqq\int_0^\infty s\,C_s^2 ds<\infty,\quad n\geqq 1.$$

Consider the remainder term $R_T^2(\lambda_T)$ (the remainder term $R_T^2(\lambda_0)$ can be evaluated in a similar way):

$$\begin{aligned}\big|R_T^2(\lambda_T)\big| &\leqq \frac{1}{T}\big|w(T)-w(n\tau)\big|\\&\quad\times\left(\sup_{0\leqq s\leqq\tau,\,x\in[\lambda_*,\lambda^*]}\big|\alpha_1(s,\,x)\big|+\sup_{i=2,\,3,\,n\tau\leqq s\leqq T,\,x\in[\lambda_*,\lambda^*]}\big|\alpha_i(s,\,x)\big|\right)\\&\quad+\frac{1}{T}\int_{n\tau}^T |w_s|\left|\frac{d}{ds}\,\alpha\big(s,\,\lambda_T(s)\big)\right| ds.\end{aligned}$$

By conditions (l)

$$\big|R_T^2(\lambda_T)\big|=\frac{o_p(1)}{\sqrt{T}}.\tag{29}$$

Now the statement of the theorem follows from estimates (21)–(29).

*Remark* 6. If $\alpha_1(s,x)$ is a linear function in $x$, then instead of condition (j) it suffices to require (d). Indeed, in this case the left-hand side of (26) is equal to zero, whereas the remaining summands in the right-hand side of (23) are evaluated by

$$\frac{O_p(1)}{\sqrt{T}}\,\|\lambda_T-\lambda_0\|_{\overline{W}^1_2}.$$

**5. Convergence of smooth functionals of the estimator.** Let $p \in (1,2)$, $q$ be its conjugate index; $K_0$ be an open bounded convex set in $\overline{W}^1_{2q}$ consisting of positive functions; and let $K$ be the closure of $K_0$ in $\overline{W}^1_{2p}$. Since $p < q$, it follows that $\overline{W}^1_{2p}$ is a compact imbedding into $\overline{W}^1_{2q}$ (Remark 1(b)), and $K$ is a compact set in $\overline{W}^1_{2p}$. Moreover, $K$ is a bounded set in $\overline{W}^1_{2q}$. Indeed, let $\sup_{f\in K_0}\int_0^\tau |f'|^{2q} ds \leqq R$. If $\{f_n\} \subset K_0$ and $f_n \to g$ in $\overline{W}^1_{2p}$, then $f_n$ converges to $g$ uniformly in $[0,\tau]$; then by the lemma [10, p. 86] being valid given the exponent $2q > 1$, the inequality $\int_0^\tau |f_n'|^{2q} ds \leqq R$ yields $\int_0^\tau |g'|^{2q} ds \leqq R$.

Suppose that the following condition is valid:

(m) The function $\lambda_0 \in K_0$.

THEOREM 4. *Let conditions* (c), (e), (g)–(i), (l), *and* (m) *be fulfilled and let the following condition be valid*:

(n1) $$\alpha_1 \in C^2\big([0,\,\tau] \times [\lambda_*, \lambda^*]\big);$$

(n2) $$\alpha_{2,\,3} \in C^2\big(\mathbf{R}^+ \times [\lambda_*, \lambda^*]\big);$$

(n3) *for* $i = 2,3$ $|\partial^2\alpha_i(s,x)/\partial s\,\partial x| \leqq D_s$, *where* $D_s$ *is bounded and*

$$\limsup_{T\to\infty} (\sqrt{T})^{-1} \int_0^T s D_s^2 ds < \infty;$$

(n4) $$\left|\frac{\partial^2\alpha_1(s,x)}{\partial s\,\partial x} - \frac{\partial^2\alpha_1(s,y)}{\partial s\,\partial x}\right| + \left|\frac{\partial^2\alpha_1(s,x)}{\partial x^2} - \frac{\partial^2\alpha_1(s,y)}{\partial x^2}\right| \leqq C\,|x-y|,$$

(n5) $$\left|\frac{\partial^2\alpha_i(s,x)}{\partial s\,\partial x} - \frac{\partial^2\alpha_i(s,y)}{\partial s\,\partial x}\right| \leqq D_s\,|x-y|, \quad i = 2,\,3,$$

*where* $D_s$ *are the same as in condition* (n3);

(n6) *for* $i = 2,3$

$$\sup_{0\leqq s\leqq\tau,\,\lambda_*\leqq x\leqq\lambda^*} \frac{1}{n}\sum_{k=0}^{n-1} \left|\frac{\partial\alpha_i(s+k\tau,x)}{\partial x}\right|^2 = O\left(\frac{1}{n^{1/4}}\right).$$

(n7) *for* $i = 2,3$, $|\partial^2\alpha_1(s,x)/\partial x^2| \leqq F_s$ *and* $F_s$ *is bounded with*

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T (sF_s^2)^P ds < \infty;$$

(n8) $$\left|\frac{\partial^2\alpha_i(s,x)}{\partial x^2} - \frac{\partial^2\alpha_i(s,y)}{\partial x^2}\right| \leqq G_s\,|x-y|, \quad i = 2,\,3,$$

*where* $G_s$ *is bounded and* $\limsup_{T\to\infty} T^{-1/(q+1)} \int_0^T (sG_s^2)^{q/(q+1)} ds < \infty$.

*Then for any function* $\varphi \in \overline{W}^1_{2q}$ *the distribution of the random variable*

$$\sqrt{\frac{T}{\tau}} \int_0^\tau \varphi(t)\left(\lambda_T(t) - \lambda_0(t)\right) dt$$

*is weakly convergent, as* $T \to \infty$, *to the normal distribution with zero mean and variance*

$$\int_0^\tau \varphi^2(t) \left( \frac{1}{\lambda_0(t)} + \left| \frac{\partial \alpha_1(t, \lambda_0)}{\partial x} \right|^2 \right)^{-1} dt.$$

*Proof.* By condition (m) there exists an $\varepsilon > 0$ such that $\lambda_0(t) \geqq \varepsilon$, $t \in [0, \tau]$. By virtue of Theorem 2 and the imbedding theorem for Sobolev spaces

$$\lim_{T\to\infty} \mathbf{P}\Big\{ \lambda_T(t) \geqq \frac{\varepsilon}{2},\ t \in [0, \tau] \Big\} = 1.$$

Further, we consider only those $T$ and $\omega$ which meet the preceding inequality. Take the Gateaux derivative $\langle Q'_T(\lambda_T), h \rangle$ of $Q_T$ at the point $\lambda_T$ along a direction $h \in \overline{W}{}_2^1[0, \tau]$, where $h$ is periodically continued with period $\tau$:

$$\begin{aligned} \big\langle Q'_T(\lambda_T),\, h \big\rangle = {} & \frac{n\tau}{T} \frac{1}{\tau} \left[ -\int_0^\tau h\, ds + \int_0^\tau \frac{\lambda_0}{\lambda_T} h\, ds + \int_0^\tau \frac{h}{\lambda_T}\, dA_n \right] \\ & - \frac{1}{T} \int_0^T \frac{\partial \alpha(s, \lambda_T)}{\partial x} \big[ \alpha(s,\, \lambda_T) - \alpha(s,\, \lambda_0) \big]\, h(s)\, ds \\ & + \frac{1}{T} \int_0^T \frac{\partial \alpha(s, \lambda_T)}{\partial x} h(s)\, dw_s + \big\langle R'_T(\lambda_T),\, h \big\rangle, \end{aligned} \tag{30}$$

where

$$\big\langle R'_T(\lambda_T),\, h \big\rangle = \frac{1}{T} \left( -\int_0^{T-n\tau} h\, ds + \int_0^{T-n\tau} \frac{h}{\lambda_T}\, dN_{n+1} \right).$$

Recall that $A_n$ and $N_n$ were introduced in section 4.

We fix a function $\psi \in \overline{W}{}_{2q}^1$. By condition (m) there exists a $\delta > 0$ such that the functions $\lambda_0 \pm \delta\psi$ belong to $K_0$. Set $K$ being convex, the functions

$$q_\pm(t) := Q_T\Big( \lambda_T + t(\lambda_0 - \lambda_T \pm \delta\psi) \Big), \quad t \in [0, 1],$$

are well defined. They attain maximum at $t = 0$, whence

$$q'_\pm(0) := \big\langle Q'_T(\lambda_T),\ \lambda_0 - \lambda_T \pm \delta\psi \big\rangle \leqq 0$$

or

$$\Big| \big\langle Q'_T(\lambda_T),\, \psi \big\rangle \Big| \leqq \frac{1}{\delta} \big\langle Q'_T(\lambda_T),\ \Delta_T \big\rangle, \tag{31}$$

where $\Delta_T$ was defined in section 4.

Let us estimate the right-hand side of (31). It is shown in [3] that under the condition of the theorem

$$\begin{aligned} \frac{n\tau}{T} \frac{1}{\tau} \left[ -\int_0^\tau \Delta_T ds + \int_0^\tau \frac{\lambda_0 \Delta_T}{\lambda_T}\, ds + \int_0^\tau \frac{\Delta_T}{\lambda_T}\, dA_n \right] &= \frac{o_p(1)}{\sqrt{T}}; \\ \big\langle R'_T(\lambda_T),\ \Delta_T \big\rangle &= \frac{o_p(1)}{T}. \end{aligned}$$

We evaluate the integral

$$I_1 := \frac{1}{T}\int_0^T \left|\alpha'_x(s,\lambda_T)\right| \left|\alpha(s,\lambda_T)-\alpha(s,\lambda_0)\right| |\Delta_T|\, ds.$$

Making use of conditions (i) and Theorem 3 gives

$$\begin{aligned} I_1 &\leqq \frac{C}{T}\int_0^T \left|\alpha'_x(s,\,\lambda_T)\right| |\Delta_T|^2 ds \leqq \frac{C(n+1)}{T}\int_0^\tau \left|\frac{\partial\alpha_1}{\partial x}(s,\,\lambda_T)\right| |\Delta_T|^2 ds \\ &+ \frac{C}{T}\int_0^T \left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}(s,\,\lambda_T)\right| |\Delta_T|^2 ds \leqq C_1\int_0^\tau |\Delta_T|^2 ds = \frac{o_p(1)}{\sqrt{T}}. \end{aligned}$$

Denote by $\Delta'_T := \lambda'_T - \lambda'_0$ the derivative of $\Delta_T$. Using conditions (n1) and (n2) and the notation $B_n$ from section 3, we evaluate the integral

$$\begin{aligned} I_2 &:= \frac{1}{T}\left|\int_0^T \alpha'_x(s,\,\lambda_T)\,\Delta_T(s)\,dw_s\right| \leqq \frac{1}{T}\left|\int_0^{n\tau} \alpha'_x(s,\,\lambda_T)\,\Delta_T\,dw_s\right| \\ &+ \frac{1}{T}\left|\int_{n\tau}^T \alpha'_x(s,\,\lambda_T)\,\Delta_T\,dw_s\right| \leqq \frac{C}{n\tau}\left|\int_0^\tau \frac{\partial\alpha_1}{\partial x}(s,\,\lambda_T)\,\Delta_T\,dB_n\right| \\ &+ \frac{1}{T}\left|\int_0^{n\tau} \frac{\partial(\alpha_2+\alpha_3)}{\partial x}(s,\,\lambda_T)\,\Delta_T\,dw_s\right| + \frac{1}{T}\left|\int_{n\tau}^T \alpha'_x(s,\,\lambda_T)\,dw_s\right| \\ &\leqq \frac{C\,|B_n(\tau)|}{n\tau}\left|\frac{\partial\alpha_1(\tau,\lambda_T(\tau))}{\partial x}\,\Delta_T(\tau)\right| + \frac{C}{n\tau}\int_0^\tau \left|B_n(s)\right| \left|\frac{d}{ds}\,\frac{\partial\alpha_1(s,\lambda_T(s))}{\partial x}\right| ds \\ &+ \frac{C}{n\tau}\int_0^\tau |B_n| \left|\frac{\partial\alpha_1}{\partial x}(s,\,\lambda_T)\,\Delta'_T\right| ds \\ &+ \frac{1}{T}\left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,\big(n\tau,\,\lambda_T(\tau)\big)\,\Delta_T(\tau)\right| \left|w(n\tau)\right| \\ &+ \frac{1}{T}\int_0^{n\tau} \left|\frac{d}{ds}\,\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_T)\right| |w_s|\,ds \\ &+ \frac{1}{T}\int_0^{n\tau} \left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_T)\,\Delta'_T\right| |w_s|\,ds + \frac{1}{T}\left|\int_{n\tau}^T \frac{\partial\alpha}{\partial x}\,(s,\,\lambda_T)\,dw_s\right|. \end{aligned} \tag{32}$$

It is not difficult to check that

$$\frac{d}{ds}\,\frac{\partial\alpha_i(s,\lambda_T)}{dx} = \frac{\partial^2\alpha_i(s,\lambda_T)}{ds\,dx} + \frac{\partial^2\alpha_i(s,\lambda_T)}{dx^2};$$

$$\frac{|B_n(\tau)|}{n\tau}\left|\frac{\partial\alpha_1}{\partial x}\,\big(\tau,\lambda_T(\tau)\big)\,\Delta_T(\tau)\right| = \frac{o_p(1)}{\sqrt{T}}\,;$$

$$\frac{1}{n\tau}\int_0^\tau |B_n| \left|\frac{\partial^2\alpha_1(s,\lambda_T)}{\partial s\,\partial x}\,\Delta_T\right| ds \leqq \frac{C}{\sqrt{n}}\left\|\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1} Y_i\right\|_{L_2[0,\tau]} \left(\int_0^\tau |\Delta_T|^2 ds\right)^{1/2} = \frac{o_p(1)}{\sqrt{T}}\,;$$

$$\frac{1}{n\tau}\int_0^\tau |B_n| \left|\frac{\partial\alpha_1(s,\lambda_T)}{\partial x}\,\Delta'_t\right| ds \leqq \frac{C}{\sqrt{n}}\left\|\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1} Y_i\right\|_{L_2[0,\tau]} \left(\int_0^\tau |\Delta'_T|^2 ds\right)^{1/2} = \frac{o_p(1)}{\sqrt{T}}\,;$$

$$\frac{1}{T}\left|w(n\tau)\right| \left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,\big(n\tau,\,\lambda_T(\tau)\big)\,\Delta_t(\tau)\right| = \frac{o_p(1)}{\sqrt{T}}.$$

By condition (n3) and Theorem 3

$$\begin{aligned}
&\frac{1}{T}\int_0^{n\tau}\left|\frac{\partial^2(\alpha_2+\alpha_3)}{\partial s\,\partial x}\,(s,\,\lambda_T)\,\Delta_T\right||w_s|\,ds \\
&\leqq \frac{1}{T}\biggl(\int_0^{n\tau} D_s^2 w_s^2 ds\biggr)^{1/2}\sqrt{n}\biggl(\int_0^{\tau}|\Delta_T|^2 ds\biggr)^{1/2} \\
&\leqq \biggl(\frac{1}{\sqrt{T}}\int_0^T D_s^2 w_s^2 ds\biggr)^{1/2}\frac{o_p(1)}{\sqrt{T}}=\frac{o_p(1)}{\sqrt{T}}, \qquad (33)
\end{aligned}$$

since

$$\limsup_{T\to\infty}\frac{1}{\sqrt{T}}\,\mathbf{E}\int_0^T D_s^2 w_s^2 ds=\limsup_{T\to\infty}\frac{1}{\sqrt{T}}\int_0^T D_s^2 s\,ds<\infty.$$

By condition (l2) we have

$$\begin{aligned}
&\frac{1}{T}\int_0^{n\tau}\left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,\Delta_T'\right||w_s|\,ds \\
&\leqq \sqrt{\frac{n}{T}}\biggl(\int_0^{\tau}|\Delta_T'|^2 ds\biggr)^{1/2}\frac{1}{\sqrt{T}}\biggl(\int_0^{n\tau} C_s^2 w_s^2 ds\biggr)^{1/2}=\frac{o_p(1)}{\sqrt{T}}.
\end{aligned}$$

"The remainder term" in the right-hand side of (32) admits estimates similar to (33); consequently, $I_2=o_p(1)/\sqrt{T}$. Thus, $|\langle Q_T'(\lambda_T),\psi\rangle|=o_p(1)/\sqrt{T}$.

Insert $\psi$ for $h$ in (30). By Theorem 3 in [3]

$$\begin{aligned}
\bigl\langle Q_T'(\lambda_T),\,\psi\bigr\rangle &= -\frac{1}{T}\int_0^T \alpha_x'(s,\,\lambda_T)\,\bigl[\alpha(s,\,\lambda_T)-\alpha(s,\lambda_0)\bigr]\,\psi\,ds \\
&\quad+\frac{1}{T}\int_0^T \alpha_x'(s,\,\lambda_T)\,\psi\,dw_s \\
&\quad+\frac{1+o(1)}{\tau}\biggl(-\int_0^{\tau}\frac{\Delta_T\psi}{\lambda_0}\,ds+\int_0^{\tau}\frac{\psi}{\lambda_0}\,dA_n\biggr)+\frac{o_p(1)}{\sqrt{T}}. \qquad (34)
\end{aligned}$$

We estimate the integral

$$\begin{aligned}
\frac{1}{T}\int_0^T \alpha_x'(s,\,\lambda_T)\,\psi\,dw_s &= \frac{1}{T}\int_0^T \alpha_x'(s,\,\lambda_0)\,\psi\,dw_s \\
&\quad+\frac{1}{T}\int_0^T \bigl[\alpha_x'(s,\,\lambda_T)-\alpha_x'(s,\,\lambda_0)\bigr]\,\psi\,dw_s := J_1+J_2.
\end{aligned}$$

Below we take $T=n\tau$; for $n\tau\leqq T<(n+1)\,\tau$ we perform the corresponding estimates of "the remainders." Then

$$J_1=\frac{1}{n\tau}\int_0^{\tau}\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_0)\,\psi\,dB_n+\frac{1}{n\tau}\int_0^{n\tau}\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_0)\,\psi\,dw_s.$$

According to the central limit theorem,

$$\frac{1}{\tau\sqrt{n}}\int_0^{\tau}\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_0)\,\psi\,dB_n\Longrightarrow\mathcal{N}\biggl(0,\,\frac{1}{\tau^2}\int_0^{\tau}\left|\frac{\partial\alpha_1}{\partial s}\,(s,\,\lambda_0)\right|^2\psi^2\,ds\biggr)$$

in distribution as $n \to \infty$. In addition, by condition (n6)

$$\begin{aligned} &\mathbf{E}\left|\frac{1}{\tau\sqrt{n}}\int_0^{n\tau}\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_0)\,\psi\,dw_s\right|^2 \\ &\quad\leqq \frac{C_1}{n}\int_0^\tau \psi^2\sum_{k=0}^{n-1}\left|\frac{\partial(\alpha_2+\alpha_3)(s+k\tau,\lambda_0(s))}{\partial x}\right|^2 ds = o(1), \quad n\to\infty. \end{aligned}$$

Thus,

$$J_1\sqrt{n}\Longrightarrow\mathcal{N}\left(0,\,\frac{1}{\tau^2}\int_0^\tau\left|\frac{\partial\alpha_1}{\partial s}\,(s,\,\lambda_0)\right|^2\psi^2\,ds\right)$$

in distribution. Evaluating $J_2$ we have

$$\begin{aligned} J_2 &\leqq \frac{1}{n\tau}\left|\int_0^\tau\left(\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_T)-\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_0)\right)\psi\,dB_n\right| \\ &\quad+\frac{1}{n\tau}\left|\int_0^{n\tau}\left(\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_T)-\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_0)\right)\psi\,dw_s\right| \\ &\leqq \frac{C|\lambda_T(\tau)-\lambda_0(\tau)|}{n\tau}\,\left|\psi(\tau)\right|\left|B_n(\tau)\right| \\ &\quad+\frac{1}{n\tau}\left(\int_0^\tau\left|\frac{d}{ds}\left(\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_T)-\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_0)\right)\right|^2\psi^2 ds\right)^{1/2}\left\|\sum_{i=0}^{n-1}Y_i\right\|_{L_2[0,\tau]} \\ &\quad+\left\|\sum_{i=0}^{n-1}Y_i\right\|_{L_2[0,\tau]}\frac{1}{n\tau}\left(\int_0^\tau|\psi'|^2\left|\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_T)-\frac{\partial\alpha_1}{\partial x}\,(s,\,\lambda_0)\right|^2 ds\right)^{1/2} \\ &\quad+\frac{C|w(n\tau)|}{n\tau}\,\left|\psi(T)\right|\left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,\big(T,\,\lambda_T(T)\big)-\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,\big(T,\,\lambda_0(T)\big)\right| \\ &\quad+\frac{1}{n\tau}\int_0^{n\tau}\left|\frac{\partial^2(\alpha_2+\alpha_3)}{\partial s\,\partial x}\,(s,\,\lambda_T)-\frac{\partial^2(\alpha_2+\alpha_3)}{\partial s\,\partial x}\,(s,\,\lambda_0)\right||\psi|\,|w_s|\,ds \\ &\quad+\frac{1}{n\tau}\int_0^{n\tau}\left|\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_T)-\frac{\partial(\alpha_2+\alpha_3)}{\partial x}\,(s,\,\lambda_0)\right||\psi'|\,|w_s|\,ds \\ &\leqq \frac{C}{\sqrt{n}}\left\|\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}Y_i\right\|_{L_2[0,\tau]}\left(\int_0^\tau|\psi'|^{2q}ds\right)^{1/(2q)}\|\lambda_T-\lambda_0\|_{\overline{W}^1_{2p}} \\ &\quad+\frac{1}{n\tau}\int_0^{n\tau}D_s\,|\Delta_T|\,|\psi|\,|w_s|\,ds+\frac{1}{n\tau}\int_0^{n\tau}C_s\,|\Delta_T|\,|\psi'|\,|w_s|\,ds+\frac{o_p(1)}{\sqrt{n}}=\frac{o_p(1)}{\sqrt{n}}. \end{aligned}$$

Let us estimate the integral

$$J_3:=\frac{1}{T}\int_0^T\alpha'_x(s,\,\lambda_T)\big[\alpha(s,\,\lambda_T)-\alpha(s,\,\lambda_0)\big]\,\psi\,ds.$$

We have

$$J_3=\frac{1}{\tau}\int_0^\tau\frac{1}{n}\sum_{i=0}^{n-1}\big|\alpha'_x(s+i\tau,\,\lambda_0)\big|^2\Delta_T\psi\,ds+\frac{o_p(1)}{\sqrt{n}}.$$

Condition (n6) and Theorem 3 yield

$$J_3 = \frac{1}{\tau}\int_0^\tau \left|\frac{\partial\alpha_1}{\partial x}(s,\,\lambda_0)\right|^2 \Delta_T\psi\,ds + r_n + \frac{o_p(1)}{\sqrt{n}},$$

where

$$|r_n| \leqq \frac{C}{T^{1/4}}\int_0^\tau |\Delta_T|\,|\psi|\,ds \leqq \frac{C}{\sqrt{T}}\left(\int_0^\tau \psi^2 ds\cdot T^{1/2}\cdot\int_0^\tau |\Delta_T|^2 ds\right)^{1/2} = \frac{o_p(1)}{\sqrt{T}}.$$

Relations (34) and (31) imply the representation

$$\sqrt{n}\,\tau J_3 + \sqrt{n}\int_0^\tau \frac{\Delta_T\psi}{\lambda_0}\,ds = \tau J_1\sqrt{n} + \sqrt{n}\int_0^\tau \frac{\psi}{\lambda_0}\,dA_n + o_p(1).$$

Hence

$$\sqrt{n}\int_0^\tau \left(\frac{1}{\lambda_0} + \left|\frac{\partial\alpha_1(s,\lambda_0)}{\partial x}\right|^2\right)\Delta_T\psi\,ds \Longrightarrow \mathcal{N}\left(0,\;\int_0^\tau \left(\frac{1}{\lambda_0} + \left|\frac{\partial\alpha_1(s,\lambda_0)}{\partial x}\right|^2\right)\psi^2 ds\right.$$

in distribution as $n\to\infty$. Letting here

$$\varphi := \left(\frac{1}{\lambda_0} + \left|\frac{\partial\alpha_1(s,\lambda_0)}{\partial x}\right|^2\right)\psi,$$

we obtain the desired relation

$$\sqrt{\frac{T}{\tau}}\int_0^\tau \varphi\Delta_T ds \Longrightarrow \mathcal{N}\left(0,\;\int_0^\tau \varphi^2\left(\frac{1}{\lambda_0} + \left|\frac{\partial\alpha_1(s,\lambda_0)}{\partial x}\right|^2\right)^{-1} ds\right).$$

Theorem 4 is proved.

**6. Comparison of estimators.** The efficiency of the estimator under consideration is measured in terms of the asymptotic variance of a smooth functional of it:

$$\sigma^2(\varphi) := \int_0^\tau \varphi^2\left(\frac{1}{\lambda_0} + \left|\frac{\partial\alpha_1(s,\lambda_0)}{\partial x}\right|^2\right)^{-1} ds.$$

If the MLE $\widetilde{\lambda}_T$ is constructed by the Poisson component $\{N_t(\lambda_0),\ 0\leqq t\leqq T\}$, then, according to [3],

$$\sqrt{\frac{T}{\tau}}\int_0^\tau \varphi(\widetilde{\lambda}_T - \lambda_0)\,ds \Longrightarrow \mathcal{N}\left(0,\;\int_0^\tau \varphi^2\lambda_0\,ds\right),$$

and the asymptotic variance is $\widetilde{\sigma}^2(\varphi) := \int_0^\tau \varphi^2\lambda_0\,ds$. If we construct the MLE $\widehat{\lambda}_T$ by observations over $\{\eta_t(\lambda_0),\ 0\leqq t\leqq T\}$, then, given

$$\frac{\partial\alpha_1(s,x)}{\partial s} \neq 0, \quad s\in[0,\,\tau],\ x\in[\lambda_*,\,\lambda^*], \tag{35}$$

we have

$$\sqrt{\frac{T}{\tau}}\int_0^\tau \varphi(\widehat{\lambda}_T - \lambda_0)\,ds \Longrightarrow \mathcal{N}\left(0,\;\int_0^\tau \varphi^2\left|\frac{\partial\alpha_1(s,\lambda_0)}{\partial x}\right|^{-2} ds\right),$$

and the asymptotic variance is

$$\widehat{\sigma}^2(\varphi) := \int_0^\tau \varphi^2 \left| \frac{\partial \alpha_1(s, \lambda_0)}{\partial x} \right|^{-2} ds.$$

Under the conditions of Theorem 4 the following inequality is valid for $\varphi \in \overline{W}^1_{2q}$, $\varphi \not\equiv 0$:

$$\sigma^2(\varphi) \leqq \widehat{\sigma}^2(\varphi), \tag{36}$$

and under the additional condition (35) $\sigma^2(\varphi) < \widetilde{\sigma}^2(\varphi)$. The strict inequality in (36) is valid if

$$\operatorname{mes} \left\{ s \in \operatorname{supp} \varphi \colon \frac{\partial \alpha_1(s, \lambda_0(s))}{\partial x} \neq 0 \right\} > 0.$$

Thus, the joint processing of the components $\{N_t(\lambda_0),\ \eta_t(\lambda_0)\}$ under observation is more efficient than their separate processing.

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